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Conformal and Lie superalgebras motivated from free fermionic fields

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Abstract

In this paper, we construct six families of conformal superalgebras of infinite type, motivated from free quadratic fermonic fields with derivatives, and we prove their simplicity. The Lie superalgebras generated by these conformal superalgebras are proven to be simple except for a few special cases in the general linear superalgebras and the type-Q lie superalgebras, in which these Lie superalgebras have a one-dimensional centre and the quotient Lie superalgebras modulo the centre are simple. Certain natural central extensions of these families of conformal superalgebras are also given. Moreover, we prove that these conformal superalgebras are generated by their finite-dimensional subspaces of minimal weight in a certain sense. It is shown that a conformal superalgebra is simple if and only if its generated Lie superalgebra does not contain a proper nontrivial ideal with a one-variable structure.

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1. Introduction

The notion of conformal superalgebras was introduced by Kac [10], as the local structure of a certain Lie superalgebra with a one-variable structure. The algebraic entity of two-dimensional quantum field theory is a certain new representation theory of Lie superalgebras with one-variable structure. In terms of vertex superalgebras, conformal superalgebras are positive parts of vertex superalgebras [10, 21], which are closed local systems in the sense of Li [15]. From the point of view of simple algebra, the category of conformal algebras is much smaller than that of vertex algebras. D'Andrea and Kac have proven that a simple conformal algebra of finite type is either isomorphic to the centreless Virasoro conformal algebra or isomorphic to a loop conformal algebra associated with a finite-dimensional simple Lie algebra [7]. In other words, there are no essentially new algebras in simple conformal algebras of finite type. For infinite type, Zel'manov has an approach of introducing certain filtration and Gel'fand–Kirillov dimensions. The approach we use here is to study simple conformal algebras of finite growth proposed by Xu [22].

In this paper, we construct six families of infinite conformal and their generated Lie superalgebras, and we prove their simplicity (cf theorems 3.1 and 4.2). The construction is motivated from free quadratic fermonic fields with derivatives. We also give certain natural central extensions of these families of conformal superalgebras, and we prove a generator theorem for each of these families of algebras (cf theorems 5.1–5.6). Besides, we define a one-variable structure on the ideals of the Lie superalgebra generated by a conformal superalgebra. Then we prove that a conformal superalgebra is simple if and only if its generated Lie superalgebra does not contain a proper nontrivial ideal with a one-variable structure. Some of our results are generalizations of Xu's results on simple conformal superalgebras of finite growth. Here, we give a more technical introduction.

Throughout this paper, the base field $\mathbb{F}$ is an arbitrary field of characteristic 0. Moreover, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{Z}^+$ denotes the set of positive integers, $\mathbb{Z}^-$ denotes the set of negative integers and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ denotes the cyclic group of order 2. When the context is clear, we use $\{0, 1\}$ to denote the elements of $\mathbb{Z}_2$. Given $m, n \in \mathbb{N}$, we also define

$$\overline{m,n} = \begin{cases} \{m, m+1, \ldots, n\} & \text{if} \quad m \leqslant n \\ \emptyset & \text{if} \quad m > n. \end{cases} \tag{1.1}$$

Before presenting the definition of conformal superalgebras, we introduce some notations. The following operator of taking residue will be used

$$\mathrm{Res}_z \left(\sum_{j\in\mathbb{Z}} \xi_j z^j \right) = \xi_{-1} \tag{1.2}$$

where ξ_j are in some vector space V. Moreover, all the binomials are assumed to be expanded in the second variable. For example,

$$\frac{1}{z-x} = \frac{1}{z(1-x/z)} = \sum_{j=0}^{\infty} z^{-1} \left(\frac{x}{z}\right)^j = \sum_{j=0}^{\infty} z^{-j-1} x^j. \tag{1.3}$$

In particular, the above equation implies

$$\mathrm{Res}_x \frac{1}{z-x} \left(\sum_{j\in\mathbb{Z}} \xi_j x^j \right) = \sum_{j=1}^{\infty} \xi_{-j} z^{-j}. \tag{1.4}$$

So the operator $\mathrm{Res}_x(1/(z-x)(\cdots))$ takes part of negative powers in a formal series and changes the variable x to z. For two vector spaces V and W, we denote by $LM(V, W)$ the space of linear maps from V to W and $V[x_1, \ldots, x_n]$ to be the set of polynomials of variables $x_1, \ldots, x_n$ over V.

A conformal superalgebra $R = R_0 \bigoplus R_1$ is a $\mathbb{Z}_2$-graded $\mathbb{F}[\partial]$-module equipped with a linear map $Y^+(\cdot, z) : R \to LM(R, R[z^{-1}]z^{-1})$ satisfying

$$Y^+(R_i, z)R_j \subset R_{i+j}[z^{-1}] \tag{1.5}$$

$$Y^+(\partial u, z) = \frac{\mathrm{d}}{\mathrm{d}z} Y^+(u, z) \tag{1.6}$$

$$Y^+(u, z)v = (-1)^{ij} \mathrm{Res}_x \frac{1}{z-x} \, \mathrm{e}^{x\partial} Y^+(v, -x)u \tag{1.7}$$

$$Y^+(u, z_1)Y^+(v, z_2) - (-1)^{ij} Y^+(v, z_2)Y^+(u, z_1) = \mathrm{Res}_x \frac{1}{z_2 - x} Y^+(Y^+(u, z_1 - x)v, x) \tag{1.8}$$

for $u \in R_i$, $v \in R_j$ with $i, j \in \mathbb{Z}_2$. $(R, \partial, Y^+(\cdot, z))$ denotes a conformal superalgebra. In section 4, we see that a conformal superalgebra with property (4.1) induces a formal distribution Lie superalgebra [10]. In that case equations (1.7) and (1.8) correspond to the skew-symmetry and Jacobi identity of Lie superalgebra respectively. Then, the central extensions in section 5 are associated with central charges.

Let $\mathcal{C}$ be any $\mathbb{Z}_2$-graded associative algebra over $\mathbb{F}$. From a $\mathbb{Z}_2$-graded tensor

$$R_i = \mathcal{C}_i \bigotimes\nolimits_{\mathbb{F}} \mathbb{F}[t_1, t_2] \qquad \text{for} \quad i \in \{0, 1\} \tag{1.9}$$

where t_1, t_2 are indeterminants. For convenience, we denote

$$u(m, n) = u \otimes t_1^m t_2^n \qquad \text{for} \quad u \in \mathcal{C} \quad m, n \in \mathbb{N}. \tag{1.10}$$

We define the $\mathbb{F}[\partial]$-action on $R = R_0 \bigoplus R_1$ by

$$\partial(u(m, n)) = (m+1)u(m+1, n) + (n+1)u(m, n+1) \tag{1.11}$$

and the structure map $Y^+(\cdot, z)$ on R by

$$\begin{aligned} Y^+(u(m_1, m_2), z)v(n_1, n_2) = {} & \binom{-n_1 - 1}{m_2} \sum_{p=0}^{m_1+m_2+n_1} \binom{p}{m_1} (uv)(p, n_2) z^{p-m_1-m_2-n_1-1} \\ & - (-1)^{ij} \binom{-n_2 - 1}{m_1} \sum_{q=0}^{m_1+m_2+n_2} \binom{q}{m_2} (vu)(n_1, q) z^{q-m_1-m_2-n_2-1} \end{aligned} \tag{1.12}$$

for $u \in \mathcal{C}_i$, $v \in \mathcal{C}_j$ with $i, j \in \mathbb{Z}_2$ and $m_1, m_2, n_1, n_2 \in \mathbb{N}$. The above formulae are motivated from quadratic free fermionic fields [21, 22]. It was verified in [21] that $(R, Y^+(\cdot, z), \partial)$ forms a conformal superalgebra.

The objective of this paper is to construct six families of simple conformal superalgebras of infinite type based on the above conformal superalgebra and to study their central extensions and the Lie superalgebras generated by them. Central extensions are related to a crucial and indispensable concept in quantum field theory called anomaly. Xu [22] constructed three families of conformal superalgebras by using mixed quadratic bosonic fermionic fields. We find that those conformal superalgebras are actually our special cases. This implies that the theory of mixed quadratic bosonic fermionic fields with derivatives can be realized by pure fermonic fields with derivatives. Since the algebras we are concerned with are infinite in dimension, it is important to know if they can be generated by interesting finite-dimensional subspace. We prove a generator theorem with this property for each family of our extended conformal superalgebras. They are analogues to those constructed by Xu in [22], in which the sets are shown to form simple Jordan algebras of types A, B and C in a certain sense.

The W_∞ algebra without a centre [3], the $W_{1+\infty}$ algebra without a centre [19] in mathematical physics and the $W_{1+\infty}(gl_k)$ algebra without a centre related to the k-component KP hierarchy studied by van de Leur [16] are all special cases of the algebras constructed in [22]. The supersymmetric analogue of $W_{1+\infty}$ [6, 25, 26] is also related to the algebras in [22], which are special cases of the algebras presented in this paper. Moreover, we deal with the issue of the central extension that is absent in [22]. This is important because Johansen [8] has shown that there exist two W_∞ (with centre) symmetries in the cohomology of the BRST operator in a twisted $N = 1$ SUSY model. To avoid pathological properties, a sensible theory often requires finite dimensionality of the graded subspaces in a representation (known as the quasi-finiteness condition). Such representations make our algebras more palatable for physical applications in a similar way as W_∞ [1, 14]. We believe that all the results in this paper will be useful in the classification of conformal field theory [12, 13, 18, 20], which is one of the most important objectives in two-dimensional quantum field theory and is related to string theory. They can also be applied to integrable dynamic systems by Xu's correspondence of conformal superalgebras to linear Hamiltonian superoperators [23].

The paper is organized as follows. In section 2, we show the relationship between the conformal superalgebra we have used (cf equation (1.12)) and the free fermionic field. The vertex algebra behind is also given. In section 3, we construct six families of classical simple Lie superalgebras over the algebra of differential operators on the Laurent polynomial algebra in one variable. In section 4, we construct six families of conformal superalgebras that generate the Lie superalgebras isomorphic to those constructed in section 3. Their simplicity is also proven. Section 5 is devoted to constructing certain natural central extensions of the Lie superalgebras and the conformal superalgebras that are constructed in sections 3 and 4. Moreover, certain generator sets of the extended conformal superalgebra are also present.

2. Free quadratic fermionic fields with derivatives

In this section, we present the conformal subalgebras related to quadratic free fermionic fields with derivatives.

Let H be a finite-dimensional vector space with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ such that there exist two subspaces H_+, H_- satisfying $H = H_+ + H_-$ and

$$\langle H_+, H_+\rangle = \langle H_-, H_-\rangle = 0. \tag{2.1}$$

Thus, H_+ is isomorphic to the dual space (H_*) through the nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$.

Let t be an indeterminate and set

$$\hat{H} = H \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]t^{1/2} \oplus \mathbb{F}\kappa \tag{2.2}$$

where κ is a symbol to denote a base vector of one-dimensional vector space. We define an algebraic operation $[\cdot,\cdot]$ on $\hat{H}$ by

$$[h_1 \otimes t^m + \lambda_1\kappa, h_2 \otimes t^n + \lambda_2\kappa] = \langle h_1, h_2\rangle \delta_{m+n,0}\kappa \tag{2.3}$$

for $h_1, h_2 \in H, m, n \in \mathbb{Z} + 1/2, \lambda_1 + \lambda_2 \in \mathbb{F}$. Then $(\hat{H}, [\cdot,\cdot])$ forms a Lie superalgebra with the $\mathbb{Z}_2$-grading

$$\hat{H}_1 = H \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]t^{1/2} \qquad \text{and} \qquad \hat{H}_0 = \mathbb{F}\kappa. \tag{2.4}$$

For convenience, we denote

$$h(m) = h \otimes t^m \qquad \text{for} \quad h \in H \quad m \in \mathbb{Z} + 1/2. \tag{2.5}$$

We set

$$\hat{H}_- = \mathrm{Span}\{h(-m) | h \in H, m \in \mathbb{N} + 1/2\} \tag{2.6}$$

$$\hat{B}_H = \mathrm{Span}\{\kappa, h(m) | h \in H, m \in \mathbb{N} + 1/2\}. \tag{2.7}$$

Then $\hat{H}_-$ and $\hat{B}_H$ are trivial Lie sub-superalgebras of $\hat{H}$ and

$$\hat{H} = \hat{H}_- \oplus \hat{B}_H. \tag{2.8}$$

Let $\mathbb{F}|0\rangle$ be a one-dimensional vector space with the base element $|0\rangle$, the vacuum state. We define an action of $\hat{B}_H$ on $|0\rangle$ by

$$h(m)|0\rangle = 0 \qquad \text{and} \qquad \kappa|0\rangle = |0\rangle \qquad \text{for} \quad h \in H \quad m \in \mathbb{N} + 1/2. \tag{2.9}$$

Then $\mathbb{F}|0\rangle$ forms a $\hat{B}_H$-module. We can view $\hat{H}_-$ and $\hat{B}_H$, respectively, as the sets of creation and annihilation operators with identity element κ of the Fock space defined thereinafter.

$U(\cdot)$ denotes the universal enveloping algebra of a Lie algebra and $\wedge(\cdot)$ denotes the exterior algebra generated by a vector space. Then the Fock space $\mathcal{F}$ can be defined as an induced $\hat{H}$-module by

$$\mathcal{F} = U(\hat{H}) \otimes_{U(\hat{B}_H)} \mathbb{F}|0\rangle (\cong \bigwedge(\hat{H}_-) \otimes_{\mathbb{F}} |0\rangle \text{ as vector spaces}). \tag{2.10}$$

Moreover, we set

$$h^+(z) = \sum_{m=0}^{\infty} h(m+1/2) z^{-m-1} \qquad h^-(z) = \sum_{m=1}^{\infty} h(-m+1/2) z^{m-1} \tag{2.11}$$

$$h(z) = h^+(z) + h^-(z) \tag{2.12}$$

for $h \in H$. As operators on $\mathcal{F}$, $\{h(z)|h \in H\}$ are called free fermionic fields.

For convenience, we denote

$$u \otimes |0\rangle = u \qquad \text{for} \quad u \in \bigwedge(\hat{H}_-). \tag{2.13}$$

Notice that the vacuum state $1_{\mathbb{F}} \otimes |0\rangle$ will then be denoted by $1_{\mathbb{F}}$.

To construct a vertex operator superalgebra, we define a linear map $Y\big(|_{\hat{R}_2}, z\big) : \mathcal{F} \to LM(\mathcal{F}, \mathcal{F}[z^{-1}, z])$ by

$$Y(1_{\mathbb{F}}, z) = \mathrm{Id}_{\mathcal{F}} \tag{2.14}$$

and

$$\begin{aligned} Y(h_1(-n_1 - 1/2)h_2(-n_2 - 1/2) \cdots h_p(-n_p - 1/2), z) \\ = \frac{1}{n!}\left(\frac{d^{n_1} h_1^-(z)}{\mathrm{d}z^{n_1}} Y(h_2(-n_2 - 1/2) \cdots h_p(-n_p - 1/2), z)\right. \\ \left. - (-1)^p Y(h_2(-n_2 - 1/2) \cdots h_p(-n_p - 1/2), z) \frac{d^{n_1} h_1^+(z)}{\mathrm{d}z^{n_1}}\right) \end{aligned} \tag{2.15}$$

for $h_1, h_2, \ldots, h_p \in H, n_1, n_2, \ldots, n_p \in \mathbb{N}$. Such a definition is actually the generalization of normal ordering of free fermionic fields, which is defined as

$$: h_1(z)h_2(z) := h_1^-(z)h_2(z) - h_2(z)h_1^+(z) \qquad \text{for} \quad h_1, h_2 \in H. \tag{2.16}$$

Let k be the dimension of H. We pick an orthonormal basis $\{b_j | j \in \overline{1, k}\}$ of H with respect to $\langle \cdot, \cdot \rangle$. Then $(\mathcal{F}, Y(\cdot, z), 1_{\mathbb{F}}, \omega)$ forms a vertex operator superalgebra [21] with the Virasoro ω element given by

$$\omega = \frac{1}{2} \sum_{j=1}^{k} b_j \left(-\frac{3}{2}\right) b_j \left(-\frac{1}{2}\right). \tag{2.17}$$

For the construction of a related conformal algebra, we set

$$\hat{R}_2 = \mathrm{Span}\{h_1(-m)h_2(-n), 1_{\mathbb{F}} | h_1 \in H_+, h_2 \in H_-, m, n \in \mathbb{N} + 1/2\}. \tag{2.18}$$

The restricted map $Y\big(|_{\hat{R}_2}, z\big) : \hat{R}_2 \to LM(\mathcal{F}, \mathcal{F}[z^{-1}, z])$ will then be equation (2.14) and

$$Y(h_1(-m - 1/2)h_2(-n - 1/2), z) = \frac{1}{m!n!}\left(\frac{\mathrm{d}^m h_1^-(z)}{\mathrm{d}z^m}\frac{\mathrm{d}^n h_2(z)}{\mathrm{d}z^n} - \frac{\mathrm{d}^n h_2(z)}{\mathrm{d}z^n}\frac{\mathrm{d}^m h_1^+(z)}{\mathrm{d}z^m}\right) \tag{2.19}$$

for $h_1 \in H_+, h_2 \in H_-$ and $m, n \in \mathbb{N}$. The operator $Y(h_1(-m - 1/2)h_2(-n - 1/2), z)$ is a quadratic fermionic field with derivatives. Moreover, we write

$$Y(u, z) = \sum_{m \in \mathbb{Z}} u_n(z)^{-n-1} \qquad Y^+(u, z) = \sum_{n=0}^{\infty} u_n(z)^{-n-1} \qquad \text{for} \quad u \in \hat{R}_2. \tag{2.20}$$

In particular,

$$\begin{aligned}(h_1(-m-1/2)h_2(-n-1/2))_k \\ = \sum_{j=0}^{\infty}\Bigg[\binom{-j-1}{-n}\binom{j+m+n-k}{m}h_1(k-m-n-j-1/2)h_2(j+1/2) \\ -\binom{-j-1}{m}\binom{j+m+n-k}{n}h_2(k-m-n-j-1/2)h_1(j+1/2)\Bigg]\end{aligned} \tag{2.21}$$

for $h_1 \in H_+, h_2 \in H_-$ and $m, n, k \in \mathbb{N}$. It can be shown that

$$Y^+(u,z)v \subset \hat{R}_2[z^{-1}]z^{-1} \qquad \text{for} \quad u, v \in \hat{R}_2. \tag{2.22}$$

Moreover, we define $\partial \in \mathrm{End}\,\hat{R}_2$ by $\partial(1_{\mathbb{F}}) = 0$ and

$$\begin{aligned}\partial(h_1(-m-1/2)h_2(-n-1/2)) = (m+1)h_1(-m-3/2)h_2(-n-1/2) \\ +(n+1)h_1(-m-1/2)h_2(-n-3/2)\end{aligned} \tag{2.23}$$

for $h_1, h_2 \in H$ and $m, n \in \mathbb{N}$. Then the family $\big(\hat{R}_2, \partial, Y^+(|_{\hat{R}_2}, z)\big)$ forms a conformal algebra.

According to linear algebra, there exists another basis $\{\varsigma_j^{\pm} \,|\, j \in I\}$ of $H_{\pm}$ [21] such that

$$\langle \varsigma_i^+, \varsigma_j^- \rangle = \delta_{i,j} \qquad \text{for} \quad i, j \in I \tag{2.24}$$

by equation (2.1) and nondegeneracy of $\langle \cdot, \cdot \rangle$, where I is an index set. By equation (2.21), we have

$$\varsigma_{j_1}^+(-m)\varsigma_{j_2}^-(m)\big(\varsigma_{j_3}^+(-m)\varsigma_{j_4}^-(-m)\big) = \delta_{j_2,j_3}\varsigma_{j_1}^+(-m)\varsigma_{j_4}^-(-m) \tag{2.25}$$

for $j_1, j_2, j_3, j_4 \in I$ and $m \in \mathbb{N}+1/2$. Expression (2.25) is essentially equivalent to matrix multiplications. This gives the motivation of the conformal algebra constructed in equation (1.12).

3. Lie superalgebras

In this section, we construct six families of Lie superalgebras from a rank-one Weyl algebra. These algebras are simple by the results in [2].

For a $\mathbb{Z}_2$-graded associative algebra $\mathcal{B} = \mathcal{B}_0 \bigoplus \mathcal{B}_1$, the associated Lie superbracket is defined by

$$[u, v] = uv - (-1)^{ij}vu \tag{3.1}$$

for $u \in \mathcal{B}_i, v \in \mathcal{B}_j$. A linear map σ on $\mathcal{B}$ is known as an involutive anti-isomorphism if

$$\sigma(\mathcal{L}_i) = \mathcal{L}_i \qquad \sigma^2 = \mathrm{Id}_{\mathcal{L}} \qquad \sigma(uv) = (-1)^{ij}\sigma(v)\sigma(u) \qquad \text{for} \quad u \in \mathcal{L}_i \quad v \in \mathcal{L}_j. \tag{3.2}$$

If we set

$$\mathcal{L}^{\sigma} = \{u \in \mathcal{L} | \sigma(u) = -u\} \tag{3.3}$$

then it will always form a Lie sub-superalgebra of $\mathcal{L}$. We let

$$\mathcal{A} = \mathbb{F}[t, t^{-1}] \qquad \partial_t = \frac{\mathrm{d}}{\mathrm{d}t}. \tag{3.4}$$

Then

$$\mathbb{A} = \sum_{n=0}^{\infty} \mathcal{A}\partial_t^n \tag{3.5}$$

forms an associative subalgebra of End $\mathcal{A}$. In fact,

$$\big(f(t)\partial_t^m\big)\big(g(t)\partial_t^n\big) = \sum_{j=0}^{m} \binom{m}{j} f(t)g^{(j)}(t)\partial_t^{m+n-j} \qquad \text{for} \quad m,n \in \mathbb{N} \quad f(t), g(t) \in \mathcal{A} \tag{3.6}$$

where

$$g^{(j)}(t) = \frac{\mathrm{d}^j g(t)}{\mathrm{d}t^j} \qquad \text{for} \quad j \in \mathbb{N}. \tag{3.7}$$

Now we are ready to construct the first two families of Lie superalgebras. For a positive integer n, we denote by $M_{n\times n}(\mathbb{A})$ the algebra of $n \times n$ matrices with entries in $\mathbb{A}$. For $i, j \in \overline{1,n}$, $E_{i,j}$ denotes the $n \times n$ matrix with 1 as its (i, j)-entry and 0 as the other entries.

We fix a positive integer k. We take nonzero polynomials

$$f_1(\partial_t), f_2(\partial_t), \ldots, f_k(\partial_t) \in \mathbb{F}[\partial_t] \tag{3.8}$$

and set

$$\mathbb{B}_p = \mathbb{A} f_p(\partial_t) \qquad \text{for} \quad p \in \overline{1,k}. \tag{3.9}$$

Then $\{\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_k\}$ is a set of nonzero left ideals of $\mathbb{A}$. We let

$$\vec{f} = (f_1, f_2, \ldots, f_k) \qquad \vec{d} = (d_1, d_2, \ldots, d_k) \qquad \text{with} \qquad d_i = \deg f_i. \tag{3.10}$$

Denote $\vec{0} = (0, 0, \ldots, 0)$.

Given $k_1 \in \overline{1,k}$, we let $k_2 = k - k_1$. Setting

$$\Delta_0 = \{1, 2, \ldots, k_1\} \qquad \text{and} \qquad \Delta_1 = \{k_1+1, k_1+2, \ldots, k\} \tag{3.11}$$

we define

$$\begin{aligned} gl(k_1, \vec{f}\,)_0 &= \sum_{i,j\in\Delta_0} \mathbb{B}_j E_{i,j} + \sum_{i,j\in\Delta_1} \mathbb{B}_j E_{i,j} \\ gl(k_1, \vec{f}\,)_1 &= \sum_{i\in\Delta_0, j\in\Delta_1} \mathbb{B}_j E_{i,j} + \mathbb{B}_i E_{j,i} \end{aligned} \tag{3.12}$$

and

$$gl(k_1, \vec{f}\,) = gl(k_1, \vec{f}\,)_0 \bigoplus gl(k_1, \vec{f}\,)_1. \tag{3.13}$$

Then $gl(k_1, \vec{f})$ forms a Lie sub-superalgebra of $M_{k\times k}(\mathbb{A})$ with the Lie superbracket defined in equation (3.1). When $\vec{d} = \vec{0}$, $gl(k_1, \vec{f}) = M_{k\times k}(\mathbb{A})$ has a one-dimensional centre $\mathbb{F}I_k$, where I_k is the $k \times k$ identity matrix. We define the quotient Lie superalgebra

$$sl(k_1, k_2; \mathbb{A}) = M_{k\times k}(\mathbb{A})/\mathbb{F}I_k. \tag{3.14}$$

Next we define

$$Q(\vec{f}) = gl(k, \vec{f}) \times gl(k, \vec{f}). \tag{3.15}$$

and an algebraic operation $\cdot$ on $Q(\vec{f})$ by

$$(a, b) \cdot (c, d) = (ac + bd, ad + bc) \qquad \text{for} \quad a, b, c, d \in gl(k, \vec{f}). \tag{3.16}$$

The $\mathbb{Z}_2$-grading of $Q(\vec{f})$ is given by

$$Q(\vec{f})_0 = (gl(k, \vec{f}), 0) \qquad Q(\vec{f})_1 = (0, gl(k, \vec{f})) \tag{3.17}$$

Then

$$Q(\vec{f}) = Q(\vec{f})_0 \bigoplus Q(\vec{f})_1. \tag{3.18}$$

forms a Lie superalgebra with the Lie superbracket defined in equation (3.1). For later use, we also denote

$$u_{[0]} = (u, 0) \qquad \text{and} \qquad u_{[1]} = (0, u) \qquad \text{for} \quad u \in gl(k, \vec{f}). \tag{3.19}$$

In order to construct the third family, we define $\tau \in \operatorname{End} \mathbb{A}$ by

$$\tau\big(h(t)\partial_t^m\big) = (-\partial_t)^m h(t) = (-1)^m \sum_{j=0}^{m} \binom{m}{j} h^{(j)}(t)\partial_t^{m-j} \qquad \text{for} \quad m \in \mathbb{N} \quad h(t) \in \mathcal{A}. \tag{3.20}$$

Then τ is an involutive anti-automorphism of the associative algebra $\mathbb{A}$, that is,

$$\tau^2 = 1 \qquad \tau(1) = 1 \qquad \tau(uv) = \tau(v)\tau(u) \qquad \text{for} \quad u, v \in \mathbb{A}. \tag{3.21}$$

For $i \in \{0, 1\}$, we define

$$(\mathbb{F}[\partial_t])_i = \sum_{n=0}^{\infty} \mathbb{F}\partial_t^{2n+i}. \tag{3.22}$$

We fix a positive integer k and $\iota \in \{0, 1\}$. Picking $k_1 \in \overline{1, k}$ and taking nonzero polynomials

$$f_1(\partial_t), f_2(\partial_t), \ldots, f_{k_1}(\partial_t) \in (\mathbb{F}[\partial_t])_\iota \qquad f_{k_1+1}(\partial_t), \ldots, f_k(\partial_t) \in (\mathbb{F}[\partial_t])_{\iota+1} \tag{3.23}$$

we set equation (3.9). For each pair $i, j \in \overline{1, k}$, we define a linear map $\rho_{i,j} : \mathbb{B}_i \to \mathbb{B}_j$ by

$$\rho_{i,j}(u f_i(\partial_t)) = (-1)^{\iota+\epsilon(i,j)}\tau(u) f_j(\partial_t) \qquad \text{for} \quad u \in \mathbb{A} \tag{3.24}$$

where

$$\epsilon(i, j) = \begin{cases} 0 & \text{if} \quad i, j \in \Delta_0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.25}$$

Then we have

$$\rho_{i,j}\rho_{j,i} = \mathrm{Id}_{\mathbb{B}_i} \qquad \text{for} \quad i, j \in \overline{1, k} \tag{3.26}$$

and

$$\rho_{p_3,p_1}(uv) = (-1)^{(i_1+i_2)(i_3+i_2)}\rho_{p_3,p_2}(v)\rho_{p_2,p_1}(u) \qquad \text{for} \quad p_j \in \Delta_{i_j} \quad u \in \mathbb{B}_{p_2} \quad v \in \mathbb{B}_{p_3}. \tag{3.27}$$

We define $gl(k_1, \vec{f})$ as in equation (3.13). Then $gl(k_1, \vec{f})$ forms a Lie sub-superalgebra of $M_{k\times k}(\mathbb{A})$. Moreover, we define a linear map $* : gl(k_1, \vec{f}) \to gl(k_1, \vec{f})$ by

$$\left(\sum_{i,j=1}^{k} u_{i,j} E_{i,j}\right)^* = \sum_{i,j=1}^{k} \rho_{i,j}(u_{j,i}) E_{i,j} \qquad \text{for} \quad u_{i,j} \in \mathbb{B}_j. \tag{3.28}$$

It can be verified that $*$ is an involutive anti-automorphism of $gl(k_1, \vec{f})$. We define

$$o(k_1, \vec{f}) = \{A \in gl(k_1, \vec{f}) | A^* = -A\} \tag{3.29}$$

with the grading inherited from $gl(k_1, \vec{f})$. Then the subspace $o(k_1, \vec{f})$ forms a Lie sub-superalgebra of $M_{k\times k}(\mathbb{A})$ with the Lie superbracket defined in equation (3.1). In fact

$$o(k_1, \vec{f}) = \operatorname{Span}\big\{t^n\partial_t^m f_j(\partial)E_{i,j} - (-1)^{\iota+\epsilon\{i,j\}}(-\partial_t)^m t^n f_i(\partial)E_{j,i} | n \in \mathbb{Z}, m \in \mathbb{N}, i, j \in \overline{1, k}\big\}. \tag{3.30}$$

Next we construct the fourth family. Taking $\vec{f}$ as equation (3.23) and setting

$$\Psi_0 = \sum_{a,b\in\{0,k\}} \left(\sum_{i,j\in\Delta_0} \mathbb{B}_j E_{i+a,j+b} + \sum_{i,j\in\Delta_1} \mathbb{B}_j E_{i+a,j+b}\right) \tag{3.31}$$

and

$$\Psi_1 = \sum_{a,b\in\{0,k\}} \left(\sum_{i\in\Delta_0, j\in\Delta_1} \mathbb{B}_j E_{i+a,j+b} + \mathbb{B}_i E_{j+a,i+b} \right) \tag{3.32}$$

then $\Psi = \Psi_0 \bigoplus \Psi_1$ forms an associative subalgebra of $M_{2k\times 2k}(\mathbb{A})$. Moreover, we define a linear map $\dagger : \Psi \to \Psi$ by

$$\left(\sum_{p,q=1}^{2k} u_{p,q} E_{p,q} \right)^{\dagger} = \sum_{i,j=1}^{k} (\rho_{i,j}(u_{k+j,k+i}) E_{i,j} + \rho_{i,j}(u_{j,i}) E_{k+i,k+j} - \rho_{i,j}(u_{j,k+i}) E_{i,k+j} - \rho_{i,j}(u_{k+j,i}) E_{k+i,j}) \tag{3.33}$$

for $u_{p,j}, u_{p,k+j} \in \mathbb{B}_j$. It can be verified that $\dagger$ is an involutive anti-automorphism of Ψ. Set

$$sp(k_1, \vec{f}) = \{A \in \Psi | A^{\dagger} = -A\} \tag{3.34}$$

with the grading inherited from Ψ. The subspace $sp(k_1, \vec{f})$ forms a Lie sub-superalgebra of $M_{2k\times 2k}(\mathbb{A})$ with the Lie superbracket defined in (3.1).

For the fifth family, we fix a $k \in Z^+$ and $\iota \in \{0, 1\}$ as before. We take nonzero polynomials

$$f_i(\partial_t) \in (\mathbb{F}[\partial_t])_{\iota} \qquad \text{for} \quad i \in \overline{1, k}. \tag{3.35}$$

For $m, n \in \overline{1, k}$, we define

$$M_{m,n}(\vec{f}) = \sum_{i\in\overline{1,m}, j\in\overline{1,n}} \mathbb{B}_j E_{i,j} \subset M_{m\times n}(\mathbb{A}). \tag{3.36}$$

We define a linear map $T : \bigcup_{m,n\in\overline{1,k}} M_{m,n}(\vec{f}) \to \bigcup_{m,n\in\overline{1,k}} M_{m,n}(\vec{f})$ such that for $m, n, i, j \in \overline{1, k}$,

$$(M_{m\times n}(\vec{f}))^T \subset M_{n\times m}(\vec{f}) \tag{3.37}$$

and

$$(u f_j(\partial_t) E_{i,j})^T = (-1)^{\iota} \tau(u) f_i(\partial_t) E_{j,i} \qquad \text{for} \quad u \in \mathbb{A}. \tag{3.38}$$

We set

$$\Psi'_0 = \sum_{i,j=1}^{k} (\mathbb{B}_j E_{i,j} + \mathbb{B}_j E_{k+i,k+j}) \tag{3.39}$$

and

$$\Psi'_1 = \sum_{i,j=1}^{k} (\mathbb{B}_j E_{i,j+k} + \mathbb{B}_j E_{i+k,j}). \tag{3.40}$$

Then $\Psi' = \Psi'_0 \bigoplus \Psi'_1$ is a $\mathbb{Z}_2$-graded associative subalgebra of $M_{2k\times 2k}(\mathbb{A})$. Moreover, we define a linear map σ_p by

$$\sigma_p \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right] = \begin{pmatrix} D^T & -B^T \\ C^T & A^T \end{pmatrix} \tag{3.41}$$

where $A, B, C, D \in M_{k,k}(\vec{f})$. It can be verified that σ_p is a supersymmetric involutive anti-automorphism of Ψ'. Setting

$$P(\vec{f}) = \{u \in \Psi' | \sigma_p(u) = -u\} \tag{3.42}$$

and defining the grading of it by

$$P(\vec{f}) = P(\vec{f})_0 \bigoplus P(\vec{f})_1 \qquad \text{with} \quad P(\vec{f})_i = P(\vec{f}) \bigcap \Psi'_i \tag{3.43}$$

then $P(\vec{f})$ forms a Lie sub-superalgebra of Ψ'.

We take $\vec{f}$ as equation (3.35) and pick $m \in \overline{1,k}$. Then the last family is constructed by setting

$$\Psi_0'' = \sum_{i,j=1}^{m} \mathbb{B}_j E_{i,j} + \sum_{i,j=1}^{k} (\mathbb{B}_j E_{m+i,m+j} + \mathbb{B}_j E_{m+k+i,m+k+j} + \mathbb{B}_j E_{k+m+i,m+j} + \mathbb{B}_j E_{m+i,k+m+j}) \tag{3.44}$$

$$\Psi_1'' = \sum_{i=1}^{m} \sum_{j=1}^{k} (\mathbb{B}_j E_{i,m+j} + \mathbb{B}_j E_{i,m+k+j} + \mathbb{B}_i E_{m+j,i} + \mathbb{B}_i E_{m+k+j,i}) \tag{3.45}$$

and then Ψ'' forms a $\mathbb{Z}_2$-graded associative subalgebra of $M_{(m+2k)\times(m+2k)}(\mathbb{A})$ with

$$\Psi'' = \Psi_0'' \bigoplus \Psi_1''. \tag{3.46}$$

Moreover, we define a linear map σ_{os} by

$$\sigma_{os}\left[\begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{1,1} & D_{1,2} \\ C_2 & D_{2,1} & D_{2,2} \end{pmatrix}\right] = \begin{pmatrix} A^T & -C_2^T & C_1^T \\ B_2^T & D_{2,2}^T & -D_{1,2}^T \\ -B_1^T & -D_{2,1}^T & D_{1,1}^T \end{pmatrix} \tag{3.47}$$

where $A \in M_{m\times m}(\vec{f})$, $B_i \in M_{m\times k}(\vec{f})$, $C_i \in M_{k\times m}(\vec{f})$, $D_{i,j} \in M_{k\times k}(\vec{f})$ for all $i, j \in \{0, 1\}$. It can be verified that σ_{os} is a supersymmetric involutive anti-automorphism of Ψ''. We define

$$osp(m, 2k; \vec{f}) = \{u \in \Psi'' | \sigma_{os}(u) = -u\} \tag{3.48}$$

and

$$osp(m, 2k; \vec{f}) = osp(m, 2k; \vec{f})_0 \bigoplus osp(m, 2k; \vec{f})_1 \tag{3.49}$$

with

$$osp(m, 2k; \vec{f})_i = osp(m, 2k; \vec{f}) \bigcap \Psi_i'' \tag{3.50}$$

forming a Lie sub-superalgebra of Ψ''.

Using the results in [2], we obtain the following theorem.

Theorem 3.1. *The Lie superalgebras $o(k_1, \vec{f})$ in equation (3.29), $sp(k_1, \vec{f})$ in equation (3.34), $P(\vec{f})$ in equation (3.42) and $osp(m, 2k; \vec{f})$ in equation (3.48) are simple. Moreover, the Lie superalgebras $gl(k_1, \vec{f})$ in equation (3.13) and $Q(\vec{f})$ in equation (3.15) are also simple provided that $\vec{d} \neq \vec{0}$. If $\vec{d} = \vec{0}$, then $sl(k_1, k_2; \mathbb{A})$ in equation (3.14) and $Q(\vec{0})/(\mathbb{F}I_k, 0)$ will be simple.*

Remark. We have proven that the conformal superalgebras $R_{[k_1,k_2],\ell+1}$, $R^*_{[k_1,k_2]}$ and $R^\dagger_{[k_1,k_2]}$ in section 4 of [22] are isomorphic special cases of $R(\vec{f})$, $R^*(\vec{f})$ and $R^\dagger(\vec{f})$, respectively. This implies that the theory of mixed free quadratic fields with derivatives (of a free bosonic field and a fermionic field) can be realized by the theory of free quadratic fermionic fields with derivatives.

4. Connections of the Lie superalgebras with the conformal superalgebras

In this section, we construct six families of simple conformal superalgebras, which generate Lie superalgebras isomorphic to those constructed in previous section.

We let $(R, Y^+(\cdot, z), \partial)$ be a conformal superalgebra such that R is a free $\mathbb{F}[\partial]$-module over its subspace V, that is,

$$R = \mathbb{F}[\partial]V \cong \mathbb{F}[\partial] \bigotimes_{\mathbb{F}} V. \tag{4.1}$$

Letting ζ be an indeterminate, we set

$$L(R) = V \bigotimes_{\mathbb{F}} \mathbb{F}[\zeta, \zeta^{-1}]. \tag{4.2}$$

For $r \in V, m \in \mathbb{N}, j \in \mathbb{Z}$, we let

$$(\partial^m v)_{[j]} = m! \binom{m-1-j}{m} v_{[j-m]} = m! \binom{m-1-j}{m} v \otimes \zeta^{-(j-m)-1}. \tag{4.3}$$

Then we can define the algebraic operation $[\cdot, \cdot]$ on $L(R)$ by

$$[u_{[a]}, v_{[b]}] = \sum_{n=0}^{\infty} \binom{a}{n} (u_n(v))_{[a+b-n]} \qquad \text{for} \quad u, v \in R \quad a, b \in \mathbb{Z} \tag{4.4}$$

where $u_n(\cdot), n \in \mathbb{N}$ are the components of $Y^+(\cdot, z)$ such that

$$Y^+(u, z) = \sum_{n=0}^{\infty} u_n z^{-n-1}. \tag{4.5}$$

Then $(L(R), [\cdot, \cdot])$ forms a Lie superalgebra [10, 21].

We call $\mathcal{I}$ an *ideal* of R if $\mathcal{I}$ is $\mathbb{F}[\partial]$-submodule of R and

$$u_n(\mathcal{I}) \subset \mathcal{I} \qquad \text{for} \quad u \in R \quad n \in \mathbb{N}. \tag{4.6}$$

A conformal superalgebra is said to be *simple* if it does not contain a proper nontrivial ideal. For a subset S of R, we define

$$\mathcal{S}_{[\mathbb{Z}]} = \mathrm{Span}\{u_{[n]} | u \in \mathcal{S}, n \in \mathbb{Z}\}. \tag{4.7}$$

An ideal J of $L(R)$ is called an *ideal with one-variable structure* if $\exists S \subset R$ such that

$$J = S_{[\mathbb{Z}]}. \tag{4.8}$$

Theorem 4.1. *A conformal superalgebra R is simple if and only if its generated Lie superalgebra $L(R)$ does not contain any proper nontrivial ideal with one-variable structure.*

Proof. Suppose $L(R)$ does not contain a proper nontrivial ideal with one-variable structure. For an ideal $\mathcal{I}$ of R, we have

$$[v_{[a]}, u_{[b]}] = \sum_{n=0}^{\infty} \binom{a}{n} (v_n(u))_{[a+b-n]} \in \mathcal{I}_{[\mathbb{Z}]} \tag{4.9}$$

with $v \in V, u \in \mathcal{I}$ and $a, b \in \mathbb{Z}$, that is, $I_{[\mathbb{Z}]}$ is an ideal of $L(R)$ with one-variable structure. Assume $\mathcal{I} \neq 0$. Then $\mathcal{I}_{[\mathbb{Z}]} = L(R)$ by the assumption of $L(R)$. This means that, given $v \in V$, we can find a finite set $\{u_m | \in \mathbb{Z}\}$ of $\mathcal{I}$ such that

$$v \otimes \zeta^{-1} = v_{[-1]} = \sum_{m \in \mathbb{Z}} u_{m[m]}. \tag{4.10}$$

It can be deduced from equation (4.3) that

$$\sum_{m \in \mathbb{N}} u_{m[m]} \in V \bigotimes_{\mathbb{F}} \mathbb{F}[\zeta] \qquad \text{and} \qquad \sum_{m \in \mathbb{Z}^-} u_{m[m]} \in V \bigotimes_{\mathbb{F}} \mathbb{F}[\zeta^{-1}]\zeta^{-1}. \tag{4.11}$$

Hence,

$$\sum_{m \in \mathbb{N}} u_{m[m]} = 0 \qquad \text{and} \qquad \sum_{m \in \mathbb{Z}^-} u_{m[m]} = \left(\sum_{m \in \mathbb{Z}^-} \frac{1}{(-m-1)!} \partial^{-m-1} u_m \right)_{[-1]} = v_{[-1]}. \tag{4.12}$$

According to equation (4.3), the map $u \to u_{[-1]}$ is an injective map from R to $L(R)$. Hence,

$$\sum_{m\in\mathbb{Z}^-} \frac{1}{(-m-1)!}\partial^{-m-1}u_m = v. \tag{4.13}$$

Since $\mathcal{I}$ is ideal of R, it is invariant under the action of ∂. Hence, $v \in \mathcal{I}$. This implies $V \subset \mathcal{I}$. Therefore, $R = \mathbb{F}[\partial]V = I$. So, R is simple.

Now we assume that R is simple. We let J be an nontrivial ideal with one-variable structure of $L(R)$. Defining

$$\mathcal{I}(J) = \{r \in R | r_{[n]} \in J \text{ for all } n \in \mathbb{Z}\} \tag{4.14}$$

then $\mathcal{I}(J)$ is a subspace of R. For $u \in \mathcal{I}(J)$ and $m \in \mathbb{Z}$, we have $(\partial u)_{[m]} = -mu_{[m-1]} \in \mathcal{J}$. So $\mathcal{I}(J)$ is a $\mathbb{F}[\partial]$-submodule of R. Let $r \in R$ and $u \in \mathcal{I}(J)$. We have

$$[r_{[0]}, u_{[b]}] = (r_0(u))_{[b]} \in J \tag{4.15}$$

for $b \in \mathbb{Z}$ since J is an ideal of $L(R)$, which means $r_0(u) \in \mathcal{I}(J)$. We now start an induction by supposing $r_m(u) \in \mathcal{I}(J)$ for all $m \leqslant k$, for some $k \in \mathbb{N}$. Note

$$[r_{[k+1]}, u_{[b]}] = (r_{k+1}(u)) + \sum_{n=0}^{k} \binom{k+1}{n} (r_n(u))_{[k+1+b-n]} \in J \tag{4.16}$$

for all $b \in \mathbb{Z}$. So $(r_{k+1}(u))_{[b]} \in J$ for all $b \in \mathbb{Z}$ by the induction assumption, which implies $r_{k+1}(u) \in \mathcal{I}(J)$ by equation (4.14). Therefore, the induction have shown that $r_b(u) \in \mathcal{I}(J)$ for $b \in \mathbb{N}$. Then $\mathcal{I}(J)$ is an nontrivial ideal of R and hence $\mathcal{I}(J) = R$. As a result, $J = L(R)$. □

Let $\mathcal{C}$ be a $\mathbb{Z}_2$-graded associative algebra over $\mathbb{F}$. We define the conformal superalgebra R by equations (1.9)–(1.12). Setting

$$V = \mathcal{C} \bigotimes_{\mathbb{F}} \mathbb{F}[t_2] = \text{Span}\{u(0,n) | u \in \mathcal{C}, n \in \mathbb{N}\} \tag{4.17}$$

it can be verified that R is a free $\mathbb{F}[\partial]$-module. We define a linear map $\nu : L(R) \to \mathbb{A} \bigotimes_{\mathbb{F}} \mathcal{C}$ by

$$\nu(m!u(0,m) \otimes \zeta^n) = t^n \partial_t^m u \qquad \text{for} \quad u \in \mathcal{C} \quad m \in \mathbb{N} \quad n \in \mathbb{Z} \tag{4.18}$$

which implies

$$\nu(u(p,m)_{[n]}) = \frac{1}{p!m!}(-\partial_t)^p t^n \partial_t^m u \qquad \text{for} \quad u \in \mathcal{C} \quad m, p \in \mathbb{N} \quad n \in \mathbb{Z}. \tag{4.19}$$

Using equation (1.12), we have

$$\begin{aligned} u(m_1, m_2)_n v(n_1, n_2) = {} & \binom{-n_1 - 1}{m_2} \binom{m_1 + m_2 + n_1 - n}{m_1} uv(m_1 + m_2 + n_1 - n, n_2) \\ & - (-1)^{ij} \binom{-n_2 - 1}{m_1} \binom{m_1 + m_2 + n_2 - n}{m_2} vu(n_1, m_1 + m_2 + n_2 - n) \end{aligned} \tag{4.20}$$

for $u \in \mathcal{C}_i$, $v \in \mathcal{C}_j$ and $n, m_1, m_2, n_1, n_2 \in \mathbb{N}$.

Besides the $\mathbb{Z}_2$-grading, we can also assign a $\mathbb{Z}$-grading to $L(R)$ by setting the grade of $u(m_1, m_2)_{[a]}$ to be $a - m_1 - m_2$ (cf equations (4.4) and (4.20)). Then, it is conceivable that our Lie superalgebras possess some quasi-finite representations [14]. This means there exists a $\mathbb{Z}$-graded $L(R)$-module $V = \bigoplus_{i\in\mathbb{Z}} V_i$ such that each V_i is finite in dimension.

Lemma 4.1. *The linear map ν is a Lie superalgebra isomorphism from $L(R)$ to $\mathbb{A}\bigotimes_{\mathbb{F}}\mathcal{C}$ with the Lie superbracket defined in equation (3.1).*

Proof. It is clear that ν is vector space isomorphism. Now we let $u \in \mathcal{C}_{g_1}, v \in \mathcal{C}_{g_2}$ with $g_1, g_2 \in \mathbb{Z}_2$, and we let $g = g_1 g_2$. For $m, r \in \mathbb{N}, n, s \in \mathbb{Z}$,

$$\begin{aligned}[]
[\nu(u(0,m)\otimes\zeta^n), \nu(v(0,r)\otimes\zeta^s)] &= \frac{1}{m!r!}\big[t^n\partial_t^m u, t^s\partial_t^r v\big] \\
&= \frac{1}{m!r!}\big(t^n\partial_t^m t^s\partial_t^r uv - (-1)^g t^s\partial_t^r t^n\partial_t^m vu\big).
\end{aligned}\tag{4.21}$$

On the other hand,

$$\begin{aligned}
\nu([u(0,m)\otimes\zeta^n, v(0,r)\otimes\zeta^s]) &= \nu\left(\sum_{i=0}^{\infty}\binom{n}{i}(u(0,m)_i v(0,r))_{[n+s-i]}\right) \\
&= \sum_{i=0}^{\infty}\frac{1}{m!r!}\left((-1)^m\binom{m,n}{i}(-\partial)^{m-i}t^{n+s-i}\partial_t^r uv\right. \\
&\quad\left. -(-1)^g\binom{m,n}{i}t^{n+s-i}\partial_t^{m+r-i}vu\right)
\end{aligned}\tag{4.22}$$

where

$$\binom{i,j}{l} = \frac{i!j!}{(i-l)!(j-l)!l!} \qquad \text{for} \quad i,j,l\in\mathbb{Z}. \tag{4.23}$$

Using

$$\partial_t^r t^n = \sum_{i=0}^{\infty}\binom{r,n}{i}t^{n-i}\partial_t^{r-i} \qquad \text{and} \qquad t^n\partial_t^m = \sum_{i=0}^{\infty}\binom{m,n}{i}(-1)^i\partial_t^{m-i}t^{n-i} \tag{4.24}$$

in (4.23), we obtain

$$\nu([u(0,m)\otimes\zeta^n, v(0,r)\otimes\zeta^s]) = [\nu(u(0,m)\otimes\zeta^n), \nu(v(0,r)\otimes\zeta^s)]. \tag{4.25}$$

□

Lemma 4.2. *Let V' be a subspace of V and $R' = \mathbb{F}[\partial]V'$. Then $(R', Y^+(\cdot,z), \partial)$ will be conformal sub-superalgebra of $(R, Y^+(\cdot,z), \partial)$ if $L(R')$ forms a Lie sub-superalgebra of $L(R)$.*

Proof. We need to show $Y^+(a,z)b \subset R'[z^{-1}]z^{-1}$ for all $a, b \in R'$. This will be done if we can show $Y^+(\partial^n u, z)\partial^m v \subset R'[z^{-1}]z^{-1}$ for all $n, m \in \mathbb{N}$ and u, v are homogeneous in V'. Now given $i, j \in \mathbb{Z}_2, u \in V'_i, v \in V'_j$ and $m, n \in \mathbb{N}$, by the definition of $Y^+(\cdot,z)$ in equations (1.6) and (1.7), we obtain

$$Y^+(\partial^n u, z)\partial^m v = (-1)^{ij}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \mathrm{Res}\frac{1}{z-x}e^{x\partial}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\sum_{c=0}^{\infty}v_c u(-x)^{-c-1}. \tag{4.26}$$

Therefore if we can show $v_c u \in R'$ for $c \in \mathbb{N}$, then we are done. Since $L(R')$ forms a Lie sub-superalgebra,

$$[v\otimes\zeta^a, u\otimes\zeta^0] = \sum_{n=0}^{\infty}\binom{a}{n}(v_n u)_{[a-n]} \in L(R') \tag{4.27}$$

for any $a \in \mathbb{N}$. Hence, by induction, $v_c u \in R'$ for all $c \in \mathbb{N}$. □

We now fix $k \in \mathbb{Z}_+$ throughout the rest of this section and define Δ_0, Δ_1 as in equation (3.11).

For the first two families of our conformal superalgebras, we let $\mathcal{C} = M_{k\times k}(\mathbb{F})$ and we define the grading on it by

$$\mathcal{C}_0 = \sum_{i,j\in\Delta_0} \mathbb{F}E_{i,j} + \sum_{i,j\in\Delta_1} \mathbb{F}E_{i,j} \qquad \text{and} \qquad \mathcal{C}_1 = \sum_{i\in\Delta_0,j\in\Delta_1} (\mathbb{F}E_{i,j} + \mathbb{F}E_{j,i}). \tag{4.28}$$

Taking $\vec{f}$ as equation (3.8), we write

$$f_p = \sum_{q=0}^{n_p} a_{p,q}\partial_t^q \qquad \text{for} \quad p \in \overline{1,k}. \tag{4.29}$$

We define

$$E_{i,j}^{\vec{f}}[m,n] = \sum_{p=0}^{n_j} m!(n+p)!a_{j,p}E_{i,j}(m,n+p) \tag{4.30}$$

and

$$E_{i,j+k}^{\vec{f}}[m,n] = \sum_{p=0}^{n_j} m!(n+p)!a_{j,p}E_{i,j+k}(m,n+p) \qquad \text{for} \quad i \in \overline{1,2k} \quad j \in \overline{1,k} \quad m,n \in \mathbb{N}. \tag{4.31}$$

Then

$$\partial\big(E_{i,j}^{\vec{f}}[m,n]\big) = E_{i,j}^{\vec{f}}[m+1,n] + E_{i,j}^{\vec{f}}[m,n+1] \qquad \text{for} \quad i,j \in \overline{1,2k} \quad m,n \in \mathbb{N}. \tag{4.32}$$

Letting $R(\vec{f}) = \mathbb{F}[\partial]V_{\vec{f}}$, where

$$V_{\vec{f}} = \text{Span}\,\big\{E_{i,j}^{\vec{f}}[0,n] | i,j \in \overline{1,k}, n \in \mathbb{N}\big\} \tag{4.33}$$

then $R(\vec{f})$ is a free $\mathbb{F}[\partial]$-module over the above $V_{\vec{f}}$. Letting ν' be $\nu|_{L(R(\vec{f}))}$, we have

$$\nu'\big(E_{i,j}^{\vec{f}}[0,m] \otimes \zeta^n\big) = t^n\partial_t^m f_j(\partial_t)E_{i,j} \qquad \text{for} \quad i,j \in \overline{1,k} \quad m \in \mathbb{N} \quad n \in \mathbb{Z}. \tag{4.34}$$

Therefore, $\nu'(L(R(\vec{f})) = gl(k_1, \vec{f})$, which implies that ν' is a Lie superalgebra isomorphism from $L(R(\vec{f}))$ to $gl(k_1, \vec{f})$ by lemma 4.1. So, $R(\vec{f})$ is a conformal superalgebra by lemma 4.2.

Next let

$$\mathcal{C} = M_{k\times k}(\mathbb{F}) \times M_{k\times k}(\mathbb{F}). \tag{4.35}$$

Denote

$$u_{[0]} = (u,0) \qquad \text{and} \qquad u_{[1]} = (0,u) \qquad \text{for} \quad u \in M_{k\times k}. \tag{4.36}$$

We define the grading on $\mathcal{C}$ by

$$\mathcal{C}_0 = (M_{k\times k}(\mathbb{F}), 0) \qquad \text{and} \qquad \mathcal{C}_1 = (0, M_{k\times k}(\mathbb{F})). \tag{4.37}$$

Setting

$$V_{\vec{f}}^Q = \text{Span}\,\big\{\big(E_{i,j}^{\vec{f}}\big)_{[p]}[0,n] | i,j \in \overline{1,k}, p \in \mathbb{Z}_2, n \in \mathbb{N}\big\} \tag{4.38}$$

we let $R^Q(\vec{f}) = \mathbb{F}[\partial]V_{\vec{f}}^Q$. It can be verified that $R^Q(\vec{f})$ is a free $\mathbb{F}[\partial]$-module over $V_{\vec{f}}^Q$. Letting ν^Q be the restriction of ν on $V_{\vec{f}}^Q$, then ν^Q is a Lie superalgebra isomorphism from $L(R^Q(\vec{f}))$ to $Q(\vec{f})$ by lemma 4.1 with

$$\nu^Q\big(\big(E_{i,j}^{\vec{f}}\big)_{[p]}[0,n] \otimes \zeta^m\big) = t^m\partial^n\big(E_{i,j}^{\vec{f}}\big)_{[p]} \qquad \text{for} \quad i,j \in \overline{1,k} \quad n \in \mathbb{N} \quad m \in Z \quad p \in \mathbb{Z}_2. \tag{4.39}$$

By lemma 4.2, $R^Q(\vec{f})$ is a conformal superalgebra.

To introduce the third family of our conformal superalgebra, we take $\vec{f}$ as equation (3.23) and we set

$$V^*_{\vec{f}} = \text{Span}\left\{E^{\vec{f}}_{i,j}[0,n] - (-1)^{\iota+\epsilon(i,j)}E^{\vec{f}}_{j,i}[n,0]\middle| i,j \in \overline{1,k}, m,n \in \mathbb{N}\right\}. \tag{4.40}$$

Let $R^*(\vec{f}) = \mathbb{F}[\partial]V^*_{\vec{f}}$. Then $R^*(\vec{f})$ is a free $\mathbb{F}[\partial]$-module over the above $V^*_{\vec{f}}$. We define a linear map ν^* to be the restriction of ν on $L(R^*(\vec{f}))$. We have

$$\nu^*\big(\big(E^{\vec{f}}_{i,j}[0,m] - (-1)^{\iota}E^{\vec{f}}_{j,i}[m,0]\big)\otimes \zeta^n\big) = t^n\partial_t^m f_j(\partial_t)E_{i,j} - (-1)^{\iota}(-\partial_t)^m t^n f_i(\partial_t)E_{j,i} \tag{4.41}$$

for $i,j \in \overline{1,k}$, $m \in \mathbb{N}$, $n \in \mathbb{Z}$. So ν^* is a Lie superalgebra isomorphism from $L(R^*(\vec{f}))$ to $o(\vec{f})$. Hence, $R^*(\vec{f}))$ is a conformal superalgebra.

Next we construct the fourth family of our conformal superalgebra. We let $\mathcal{C}$ be $M_{2k\times 2k}(\mathbb{F})$ and we define the grading by

$$\mathcal{C}_0 = \sum_{a,b\in\{0,k\}}\left(\sum_{i,j\in\Delta_0}\mathbb{F}E_{i+a,j+b} + \sum_{i,j\in\Delta_1}\mathbb{F}_j E_{i+a,j+b}\right) \tag{4.42}$$

and

$$\mathcal{C}_1 = \sum_{a,b\in\{0,k\}}\left(\sum_{i\in\Delta_0,j\in\Delta_1}\mathbb{F}E_{i+a,j+b} + \mathbb{F}E_{j+a,i+b}\right). \tag{4.43}$$

Take $\vec{f}$ as (3.23) and set

$$V^{\dagger}_{\vec{f}} = \text{Span}\big\{E^{\vec{f}}_{i,j}[0,n] - (-1)^{\iota+\epsilon\{i,j\}}E^{\vec{f}}_{k+j,k+i}[n,0], E^{\vec{f}}_{i,k+j}[0,n] + (-1)^{\iota+\epsilon\{i,j\}}E^{\vec{f}}_{j,k+i}[n,0],$$
$$E^{\vec{f}}_{k+i,j}[0,n] + (-1)^{\iota+\epsilon\{i,j\}}E^{\vec{f}}_{k+j,i}[n,0]| i,j \in \overline{1,k}, n \in \mathbb{N}\big\}. \tag{4.44}$$

We define $R^{\dagger}(\vec{f}) = \mathbb{F}[\partial]V^{\dagger}_{\vec{f}}$. Then $R^{\dagger}(\vec{f})$ is a free $\mathbb{F}[\partial]$-module over the above $V^{\dagger}_{\vec{f}}$. Letting $\nu^{\dagger}$ be the restriction of ν on $L(R^{\dagger}(\vec{f}))$, then $\nu^{\dagger}$ is surjective and, hence, $\nu^{\dagger}$ is a Lie superalgebra isomorphism from $L(R^{\dagger}(\vec{f}))$ to $sp(k_1,\vec{f})$. By lemma 4.2, $L(R^{\dagger}(\vec{f}))$ is a conformal superalgebra.

For the construction of the fifth family of our conformal superalgebra, we let $\mathcal{C}$ be $M_{2k\times 2k}(\mathbb{F})$ and we define its grading by

$$\mathcal{C}_0 = \sum_{i,j=1}^{k}(\mathbb{F}E_{i,j} + \mathbb{F}E_{i+k,j+k}) \qquad \text{and} \qquad \mathcal{C}_1 = \sum_{i,j=1}^{k}(\mathbb{F}E_{i+k,j} + \mathbb{F}E_{i,j+k}). \tag{4.45}$$

Taking $\vec{f}$ as equation (3.35) and setting

$$V^P_{\vec{f}} = \text{Span}\big\{E^{\vec{f}}_{i,j}[0,n] - (-1)^{\iota}E^{\vec{f}}_{k+j,k+i}[n,0], E^{\vec{f}}_{i,k+j}[0,n] + (-1)^{\iota}E^{\vec{f}}_{j,k+i}[n,0],$$
$$E^{\vec{f}}_{k+i,j}[0,n] - (-1)^{\iota}E^{\vec{f}}_{k+j,i}[n,0]| i,j \in \overline{1,k}, n \in \mathbb{N}\big\} \tag{4.46}$$

then $R^P(\vec{f})$ is a free $\mathbb{F}[\partial]$-module over the above $V^P(\vec{f})$. Letting ν^P be the restriction of ν on $L(R^P(\vec{f}))$, then ν^P is surjective and, hence, ν^P is a Lie superalgebra isomorphism from $L(R^P(\vec{f}))$ to $P(\vec{f})$. By lemma 4.2, $R^P(\vec{f})$ is a conformal superalgebra.

For the last family, we pick $m \in \overline{1,k}$. Letting $\mathcal{C}$ be $M_{(m+2k)\times(m+2k)}(\mathbb{F})$ with the grading by

$$\mathcal{C}_0 = \sum_{i,j\in\overline{1,m}}\mathbb{F}E_{i,j} + \sum_{i,j\in\overline{m+1,m+2k}}\mathbb{F}E_{i,j} \qquad \text{and} \qquad \mathcal{C}_1 = \sum_{i\in\overline{1,m},\, j\in\overline{m+1,m+2k}}(\mathbb{F}E_{i,j} + \mathbb{F}E_{j,i}) \tag{4.47}$$

we take $\vec{f}$ as equation (3.35) and we let

$$\begin{aligned} V^{osp}_{\vec{f}} = \mathrm{Span}\big\{ & E^{\vec{f}}_{a,b}[0,r] - (-1)^{\iota} E^{\vec{f}}_{b,a}[r,0],\, E^{\vec{f}}_{a,j+m}[0,r] + (-1)^{\iota} E^{\vec{f}}_{j+m+k,a}[r,0], \\ & E^{\vec{f}}_{a,j+m+k}[0,r] - (-1)^{\iota} E^{\vec{f}}_{j+m,a}[r,0],\, E^{\vec{f}}_{i+m,j+m}[0,r] - (-1)^{\iota} E^{\vec{f}}_{j+m+k,i+m+k}[r,0], \\ & E^{\vec{f}}_{i+m,j+m+k}[0,r] + (-1)^{\iota} E^{\vec{f}}_{j+m,i+m+k}[r,0],\, E^{\vec{f}}_{i+m+k,j+m}[0,r] \\ & + (-1)^{\iota} E^{\vec{f}}_{j+m+k,i+m}[r,0], \mid n \in \mathbb{Z}, r \in \mathbb{N}, i, j \in \overline{1,k}, a, b \in \overline{1,m}\big\}. \end{aligned} \tag{4.48}$$

We define $R^{osp}(\vec{f}) = \mathbb{F}[\partial]V^{osp}(\vec{f})$. Then $R^{osp}(\vec{f})$ is a free $\mathbb{F}[\partial]$-module over the above $V^{osp}_{\vec{f}}$. Letting ν^{osp} be the restriction of ν on $L(R^{osp}(\vec{f}))$, it can be shown that ν^{osp} is surjective. As a result, ν^{osp} is a Lie superalgebra isomorphism from $L(R^{osp}(\vec{f}))$ to $osp(m, 2k; \vec{f})$. Hence, $R^{osp}(\vec{f})$ is a conformal superalgebra by lemma 4.2.

Theorem 4.2. *The conformal superalgebras* $R(\vec{f}), R^{Q}(\vec{f}), R^{*}(\vec{f}), R^{\dagger}(\vec{f}), R^{P}(\vec{f})$ *and* $R^{osp}(\vec{f})$ *are all simple.*

Proof. Since $L(R^{*}(\vec{f})), L(R^{\dagger}(\vec{f})), L(R^{P}(\vec{f}))$ and $L(R^{osp}(\vec{f}))$ are all simple, and hence do not contain proper nontrivial ideal (with one variable structure), theorem 4.1 implies that $R^{*}(\vec{f}), R^{\dagger}(\vec{f}), R^{P}(\vec{f})$ and $R^{osp}(\vec{f})$ are all simple. By the same reason $R(\vec{f})$ and $R^{Q}(\vec{f})$ are simple provided that $\vec{d} \neq \vec{0}$. For $\vec{d} = \vec{0}$, the only proper nontrivial ideal of $gl(k_1, \vec{f})$ and $Q(\vec{f})$ are $\mathbb{F}I_k$ and $(\mathbb{F}I_k, 0)$, respectively. Hence $\mathbb{F}I_k(0,0) \otimes \zeta^0$ and $\mathbb{F}(I_k(0,0), 0) \otimes \zeta^0$ will be the only nontrivial ideal for $L(R(\vec{f}))$ and $L(Q(\vec{f}))$, respectively. Let $\mathcal{I}$ be a nontrivial ideal of $R(\vec{f})$. Then $\mathcal{I}_{[\mathbb{Z}]}$ is a nontrivial ideal of $L(R(\vec{f}))$ by equation (4.9). Therefore, $\mathcal{I}_{\mathbb{Z}}$ is either equal to $\mathbb{F}I_k(0,0) \otimes \zeta^0$ or $L(R(\vec{f}))$. Since it is not equal to the former, we have $\mathcal{I}_{[\mathbb{Z}]} = L(R(\vec{f}))$. We can use equations (4.10)–(4.13) to show that $\mathcal{I} = R(\vec{f})$. Therefore $R(\vec{f})$ is simple. Similar arguments show that $Q(\vec{f})$ is also simple. □

5. Central extensions and generators

In this section, we construct certain central extensions of the conformal superalgebras constructed in section 4 and the Lie superalgebras constructed in section 3. Moreover, we give certain generator sets of the extended conformal superalgebras.

Let $\mathcal{C}$ and A be a $\mathbb{Z}_2$-graded associative algebra and an associative algebras, respectively, and over the same field $\mathbb{F}$. We define

$$\mathcal{B} = \mathcal{C} \bigotimes\nolimits_{\mathbb{F}} A \tag{5.1}$$

to be the $\mathbb{Z}_2$-graded tensor algebra, that is,

$$(u_1 \otimes v_1)(u_2 \otimes v_2) = u_1 u_2 \otimes v_1 v_2 \qquad \text{for} \quad u_1, u_2 \in \mathcal{C} \quad v_1, v_2 \in A \tag{5.2}$$

and $\mathcal{B}$ inherits the grading of $\mathcal{C}$. Then $\mathcal{B}$ forms a Lie superalgebra with the Lie superbracket defined in equation (3.1). Suppose that $\theta(\cdot,\cdot) : A \times A \to \mathbb{F}$ is a 2-cocycle of A, that is,

$$\theta(u, v) = -\theta(v, u) \qquad \theta(uv, w) + \theta(vw, u) + \theta(wu, v) = 0 \tag{5.3}$$

for $u, v, w \in A$. We let $\kappa(\cdot,\cdot) : \mathcal{C} \times \mathcal{C} \to \mathbb{F}$ be a supersymmetric associative bilinear form, that is,

$$\kappa(u, v) = (-1)^{ij}\kappa(v, u) \qquad \text{for} \quad u \in \mathcal{C}_i \quad v \in \mathcal{B}_j \quad i, j \in \mathbb{Z}_2 \tag{5.4}$$

and

$$\kappa(uv, w) = \kappa(u, vw) \qquad \text{for} \quad u, v, w \in \mathcal{B}. \tag{5.5}$$

We say that $\kappa(\cdot,\cdot)$ is *graded* with *grading* $g \in \mathbb{Z}_2$ if

$$\kappa(\mathcal{C}_i, \mathcal{C}_j) = 0 \qquad \text{for} \quad ij = g + 1 \tag{5.6}$$

where $i, j \in \mathbb{Z}_2$. Now we define a bilinear form $\vartheta(\cdot,\cdot) : \mathcal{B} \times \mathcal{B} \to \mathbb{F}$ by

$$\vartheta(u_1 \otimes v_1, u_2 \otimes v_2) = \kappa(u_1, u_2)\theta(v_1, v_2) \qquad \text{for} \quad u_1, u_2 \in \mathcal{C} \quad v_1, v_2 \in A. \tag{5.7}$$

A bilinear form Ω of a Lie superalgebra L is said to be a *2-cocycle* of L if

$$\Omega(u, v) = -(-1)^{ij}\Omega(v, u) \tag{5.8}$$

and

$$\Omega([u, v], w) + (-1)^{(r+j)i}\Omega([v, w], u) + (-1)^{r(i+j)}\Omega([w, u], v) = 0 \tag{5.9}$$

for $u \in L_i$, $v \in L_j$ and $w \in L_r$, where $i, j, k \in \mathbb{Z}_2$. We call a 2-cocycle Ω of Lie superalgebra L a *graded 2-cocycle* with *grading* $g \in Z_2$ if $i, j \in \mathbb{Z}_2$, then

$$\Omega(L_i, L_j) = 0 \qquad \text{when} \quad ij = g + 1. \tag{5.10}$$

Lemma 5.1. *The bilinear form $\vartheta(\cdot,\cdot)$ is a 2-cocycle of the Lie superalgebra $\mathcal{B}$. Moreover, if $\kappa(\cdot,\cdot)$ is graded, then $\vartheta(\cdot,\cdot)$ is also graded with the same grading as $\kappa(\cdot,\cdot)$.*

Proof. This can be seen by checking equations (5.8)–(5.10) directly. □

The central extension of a Lie superalgebra $(\mathcal{G}, [\cdot,\cdot])$ associated with graded 2-cocycle ϑ is the vector space

$$\hat{\mathcal{G}} = \mathcal{G} \bigoplus \mathbb{F}c \tag{5.11}$$

with the Lie superbracket defined by

$$[u + \lambda c, v + \mu c\hat{]} = [u, v] + \vartheta(u, v)c \qquad \text{for} \quad u, v \in \mathcal{G} \quad \lambda, \mu \in \mathbb{F} \tag{5.12}$$

where c is a symbol for the base element of one-dimensional space $\mathbb{F}c$. The grading of c can be defined as the same as the grading of ϑ. The properties (5.8) and (5.9) of ϑ imply that the Lie bracket (5.12) satisfies the super skew-symmetry and super Jacobian identity.

Now putting $A = \mathbb{A}$ into equation (5.1), hence $\mathcal{B} = \mathcal{C} \bigotimes_{\mathbb{F}} \mathbb{A}$. We take $\theta(\cdot,\cdot)$ to be the 2-cocycle determined by Li [17], i.e.

$$\theta\left(t^n \partial_t^m, t^s \partial_t^r\right) = \delta_{n-m+s-r,0}(-1)^m m! r! \binom{n}{m + r + 1} \tag{5.13}$$

for $n, s \in \mathbb{Z}$, $m, r \in \mathbb{N}$. Then for a given graded supersymmetric associative bilinear form κ of $\mathcal{C}$, we can have central extension of $\mathcal{B}$ by previous argument. Moreover, the Lie superbracket on $\hat{\mathcal{B}}$ is given by

$$\left[ut^n \partial_t^m + \lambda c, vt^s \partial_t^r + \mu c\hat{\right]} = \left[ut^n \partial_t^m, vt^s \partial_t^r\right] + \kappa(u, v)\delta_{n-m+s-r,0}(-1)^m m! r! \binom{n}{m + r + 1} c \tag{5.14}$$

where $u, v \in \mathcal{C}$, $n, s \in \mathbb{Z}$, $m, r \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{F}$.

Next we consider the central extensions of the conformal superalgebra $R = \mathcal{C} \otimes_{\mathbb{F}} \mathbb{F}[t_1, t_2]$ (cf equations (1.9) to (1.12)) with graded supersymmetric associative bilinear form $\kappa(\cdot,\cdot)$ of $\mathcal{C}$. We define

$$\hat{R} = R \bigoplus \mathbb{F}\mathbf{1} \tag{5.15}$$

where $\mathbf{1}$ is a symbol for a base element of one-dimensional space $\mathbb{F}\mathbf{1}$. We define the $\mathbb{F}[\partial]$-action on R by equation (1.11) and $\partial\mathbf{1} = 0$. The grading of $\mathbf{1}$ is defined to be the same as the grading of κ.

Moreover, the structure map $Y^+(\cdot, z)$ on $\hat{R}$ is defined by

$$Y^+(w, z)\mathbf{1} = Y^+(\mathbf{1}, z)w = 0 \qquad \text{for} \quad w \in \hat{R} \tag{5.16}$$

and

$$\begin{aligned} u(m_1, m_2)_n v(n_1, n_2) &= \binom{-n_1 - 1}{m_2}\binom{m_1 + m_2 + n_1 - n}{m_1} uv(m_1 + m_2 + n_1 - n, n_2) \\ &\quad - (-1)^{ij}\binom{-n_2 - 1}{m_1}\binom{m_1 + m_2 + n_2 - n}{m_2} vu(n_1, m_1 + m_2 + n_2 - n) \\ &\quad + \delta_{n, m_1+m_2+n_1+n_2+1}\binom{-n_1 - 1}{m_2}\binom{-n_2 - 1}{m_1}\kappa(u, v)\mathbf{1} \end{aligned} \tag{5.17}$$

for $u \in \mathcal{C}_i$, $v \in \mathcal{C}_j$, $i, j \in \mathbb{Z}_2$ and $m_1, m_2, n_1, n_2 \in \mathbb{N}$. It can be verified that $\hat{R}$ is a conformal superalgebra [21]. We define

$$L(\hat{R}) = L(R) \bigoplus \mathbb{F}\mathbf{1} \tag{5.18}$$

and

$$\mathbf{1}(z) = \mathbf{1}. \tag{5.19}$$

Then we can use equation (4.4) to define the Lie superbracket on $L(\hat{R})$. Explicitly,

$$\begin{aligned} [u(0, m) \otimes \zeta^n + \lambda\mathbf{1}, v(0, r) \otimes \zeta^s + \mu\mathbf{1}\hat{]} &= [u(0, m) \otimes \zeta^n, v(0, r) \otimes \zeta^s] \\ &\quad + (-1)^m\binom{n}{m + r + 1}\kappa(u, v)\delta_{n+s-m-r,0}\mathbf{1} \end{aligned} \tag{5.20}$$

where $u, v \in \mathcal{C}$, $n, r \in \mathbb{Z}$, $m, s \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{F}$. We define a linear map $\hat{\nu} : L(\hat{R}) \to \hat{L}(R)$ such that

$$\hat{\nu}(u(0, m) \otimes \zeta^n + \mathbf{1}) = ut^n\partial_t^m + c \tag{5.21}$$

where $n, s \in \mathbb{Z}$, $m \in \mathbb{N}$, and $\hat{L}(R)$ is the central extension of $L(R)$ with Lie superbracket defined according to equation (5.14).

Now we fix $m, r \in \mathbb{N}$, $n, s, \in \mathbb{Z}$, $\lambda, \mu \in \mathbb{F}$ and $u, v \in \mathcal{C}$. We have

$$\begin{aligned} \hat{\nu}([u(0, m) \otimes \zeta^n + \lambda\mathbf{1}, v(0, r) \otimes \zeta^s + \mu\mathbf{1}\hat{]}) &= \nu[u(0, m) \otimes \zeta^n + v(0, r) \otimes \zeta^s] \\ &\quad + (-1)^m\binom{n}{m + r + 1}\kappa(u, v)\delta_{n+s-m-r,0}\, c. \end{aligned} \tag{5.22}$$

On the other hand

$$\begin{aligned} &[\hat{\nu}(u(0, m) \otimes \zeta^n + \lambda\mathbf{1}), \hat{\nu}(v(0, r) \otimes \zeta^s + \mu\mathbf{1})\hat{]} \\ &\quad = \left[ut^n\partial_t^m, vt^s\partial_t^r\right] + (-1)^m\binom{n}{m + r + 1}\kappa(u, v)\delta_{n+s-m-r,0}\, c. \end{aligned} \tag{5.23}$$

Since we have already shown that ν is an isomorphism of $L(R)$ in lemma 3.1, we have the following lemma.

Lemma 5.2. *The two Lie superalgebras $L(\hat{R})$ and $\hat{L}(R)$ are isomorphic to each other.*

Now we are ready to give the central extension of our conformal superalgebras $R(\vec{f})$, $R^Q(\vec{f})$, $R^*(\vec{f})$, $R^\dagger(\vec{f})$, $R^P(\vec{f})$ and $R^{osp}(\vec{f})$. At the same time, we give a set of generators for them. For a subset S of a conformal superalgebra R, we define

$$\mathrm{C}(S) = \mathrm{Span}\left\{u^1_{m_1}\ldots u^p_{m_p}v | u^j, v \in S, p, m_j \in \mathbb{N}\right\}. \tag{5.24}$$

We say that an element or a subset of R can be *generated by* S if they belong to $\mathrm{C}(S)$. If the whole R can be generated by S, then S will be called a *set of generators of* R.

Recall that the conformal superalgebra $R = M_{k\times k}(\mathbb{A})\bigotimes_{\mathbb{F}}\mathbb{F}[t_1, t_2]$ contains $R(\vec{f})$ and $R^*(\vec{f})$ as two conformal sub-superalgebras. We define the bilinear form κ of $M_{k\times k}(\mathbb{A})$ by

$$\kappa(A, B) = str(AB) \qquad \text{for} \quad A, B \in M_{k\times k}(\mathbb{F}) \tag{5.25}$$

where $str : M_{k\times k}(\mathbb{F}) \to \mathbb{F}$ is linear map defined by

$$str(E_{i,j}) = (-1)^{d(i)}\delta_{i,j} \qquad \text{for} \quad i, j \in \overline{1, k}. \tag{5.26}$$

It can be shown that κ is supersymmetric associative bilinear with 0 grading.

We pick $\vec{f}$ as equation (3.8), and we define

$$\hat{R}(\vec{f}) = R(\vec{f})\bigoplus \mathbb{F}\mathbf{1} = \mathrm{Span}\big\{E^{\vec{f}}_{i,j}[m, n], \mathbf{1}|i, j \in \overline{1, k}, m, n \in \mathbb{N}\big\}. \tag{5.27}$$

Then

$$\begin{aligned} E^{\vec{f}}_{i,j}[p, m]_n E^{\vec{f}}_{c,d}[q, r] &= \sum_{u=0}^{n_j}(-1)^{m+u}\frac{(m+u+q)!}{(m+u+q-n)!}a_{j,u}\delta_{j,c}E^{\vec{f}}_{i,d}[p+m+u+q-n, r] \\ &\quad - (-1)^G\sum_{v=0}^{n_d}(-1)^p\frac{(r+v+p)!}{(r+v+p-n)!}a_{d,v}\delta_{d,i}E^{\vec{f}}_{c,j}[q, p+m+r+v-n] \\ &\quad + \sum_{u=0}^{n_j}\sum_{v=0}^{n_d}(q+m+u)!(r+v+p)!(-1)^{d(i)+m+u+p}a_{j,u}a_{d,v}\delta_{j,c}\delta_{i,d}\delta_{n,p+m+q+r+v+1}\mathbf{1} \end{aligned} \tag{5.28}$$

where $i, j, c, d \in \overline{1, k}$, $G = (d(i) + d(j))(d(c) + d(d))$ and $p, m, n, c, d \in \mathbb{N}$.

In the rest of the paper, we let $k = k_1 + k_2$ with $k_1, k_2 \in \mathbb{Z}^+$. We define

$$S_{\vec{f}} = \big\{E^{\vec{f}}_{i,j}[0, 0], E^{\vec{f}}_{j,i}[0, 0]|i \in \Delta_0, j \in \Delta_1\big\}. \tag{5.29}$$

Theorem 5.1. *When $k > 2$ and $\vec{d} \neq \vec{0}$, the conformal superalgebra $\hat{R}(\vec{f})$ can be generated by $S_{\vec{f}}$. For $k = 2$ or $\vec{d} = \vec{0}$, $\hat{R}(\vec{f})$ can be generated by $S_{\vec{f}}\bigcup\big\{E^{\vec{f}}_{k,k}[0, 1]\big\}$.*

Proof. By

$$E^{\vec{f}}_{k,1}[0, 0]_{n_1+n_k+1}E^{\vec{f}}_{1,k}[0, 0] = (-1)^{1+n_k}n_1!n_k!a_{1,n_1}a_{k,n_k}\mathbf{1} \tag{5.30}$$

we know $\mathbf{1}$ can be generated by $S_{\vec{f}}$ and we can ignore the extension part in equation (5.28) for the rest of our proof. □

Claim: If $E^{\vec{f}}_{a,b}[\mathbb{N}, \overline{0, r}]$ can be generated by $S_{\vec{f}}$ or by $S_{\vec{f}}\bigcup\big\{E^{\vec{f}}_{k,k}[0, 1]\big\}$ for $r \in \mathbb{N}$, $a, b \in \overline{1, k}$ with $d(a) \neq d(b)$, then $E^{\vec{f}}_{b,b}[\mathbb{N}, \overline{0, r}]$, $E^{\vec{f}}_{b,a}[\mathbb{N}, \overline{0, r}]$ and $E^{\vec{f}}_{a,a}[\mathbb{N}, \overline{0, r}]$ can also be generated by $S_{\vec{f}}$ or by $S_{\vec{f}}\bigcup\big\{E^{\vec{f}}_{k,k}[0, 1]\big\}$, respectively.

To prove the claim, we employ its hypothesis. Note that

$$\begin{aligned} &E^{\vec{f}}_{b,a}[0, 0]_{(n_a+n_b+r'+1)}E^{\vec{f}}_{a,b}[q+n_b+r'+1, r'] \\ &\qquad = \sum_{u=0}^{n_a}(-1)^u\frac{(u+q+n_b+r'+1)!}{(u+q-n_a)!}a_{a,u}E^{\vec{f}}_{b,b}[u+q-n_a, r'] \end{aligned} \tag{5.31}$$

for $q, r' \in \mathbb{N}$. Hence, we can use induction on q for different $r' \leqslant r$ to show that $E^{\vec{f}}_{b,b}[\mathbb{N}, \overline{0,r}]$ can be generated by $S_{\vec{f}}$ or by $S_{\vec{f}} \bigcup \left\{E^{\vec{f}}_{k,k}[0,1]\right\}$, respectively. Then using

$$E^{\vec{f}}_{b,a}[0,0]_{n_b} E^{\vec{f}}_{b,b}[q,r'] = -\sum_{v=0}^{n_b} a_{b,v} \frac{(r+v)!}{(r+v-n_b)!} E^{\vec{f}}_{b,a}[q, r'+v-n_b] \quad (5.32)$$

for $q, r' \in \mathbb{N}$, we can carry out induction on $r' \leqslant r$ for different $q \in \mathbb{N}$ to show that $E^{\vec{f}}_{b,a}[\mathbb{N}, \overline{0,r}]$ can be generated by $S_{\vec{f}}$ or by $S_{\vec{f}} \bigcup \left\{E^{\vec{f}}_{k,k}[0,1]\right\}$, respectively. This implies that $E^{\vec{f}}_{b,a}[\mathbb{N}, \overline{0,r}]$ can also be generated by $S_{\vec{f}}$ or by $S_{\vec{f}} \bigcup \left\{E^{\vec{f}}_{k,k}[0,1]\right\}$, respectively.

Now we can divide the proof into two cases:

Case: $k > 2$and$\vec{d} \neq \vec{0}$

There exists $j \in \overline{1,k}$ such that $n_j \neq 0$. We pick $i \in \Delta_{d(j)+1}$, then for $s \in \overline{1,k} - \{i, j\} \neq \emptyset$, we have

$$E^{\vec{f}}_{i,j}[0,0]_{n_j-\epsilon} E^{\vec{f}}_{j,s}[q,r] = \sum_{u=0}^{n_j} (-1)^u \frac{(u+q)!}{(u+q-n_j+\epsilon)!} a_{j,u} E^{\vec{f}}_{i,s}[u+q-n_j+\epsilon, r] \quad (5.33)$$

and

$$E^{\vec{f}}_{j,i}[0,0]_{n_i} E^{\vec{f}}_{i,s}[q,r] = \sum_{u=0}^{n_i} (-1)^u \frac{(u+q)!}{(u+q-n_i)!} a_{i,u} E^{\vec{f}}_{j,s}[u+q-n_j, r] \quad (5.34)$$

where $q, r \in \mathbb{N}$ and $\epsilon \in \{0,1\}$. Given $r \in \mathbb{N}$, equation (5.33) with $\epsilon = 0$ tells us that if $E^{\vec{f}}_{j,s}[0,r]$ can be generated by $S_{\vec{f}}$, then so does $E^{\vec{f}}_{i,s}[0,r]$ while equation (5.34) implies the opposite. Since either $E^{\vec{f}}_{j,s}[0,0]$ or $E^{\vec{f}}_{i,s}[0,0]$ is in $S_{\vec{f}}$, both can be generated by $S_{\vec{f}}$. We suppose that $E^{\vec{f}}_{j,s}[q',r]$ and $E^{\vec{f}}_{i,s}[q',r]$ can be generated by $S_{\vec{f}}$ for all $q' \leqslant q$ with a $q \in \mathbb{N}$. Equation (5.33) with $\epsilon = 1$ implies $E^{\vec{f}}_{i,s}[q+1,r]$ can be generated by $S_{\vec{f}}$ and then equation (5.34) implies that $E^{\vec{f}}_{j,s}[q+1,r]$ can also be generated by $S_{\vec{f}}$. In particular, $E^{\vec{f}}_{j,s}[l,0]$ and $E^{\vec{f}}_{i,s}[l,0]$ can be generated by $S_{\vec{f}}$ for all $l \in \mathbb{N}$ by induction.

Next we would like to prove that $E^{\vec{f}}_{s,i}[l,0]$ can be generated by $S_{\vec{f}}$ for all $l \in \mathbb{N}$. If $d(s) = d(i)$, then $E^{\vec{f}}_{s,i}[\mathbb{N},0]$ can be generated by $S_{\vec{f}}$ because of the symmetry between i and s. On the other hand, if $d(s) = d(j)$, then $E^{\vec{f}}_{i,s}[\mathbb{N},0]$ can be generated by $S_{\vec{f}}$ implying that $E^{\vec{f}}_{s,i}[\mathbb{N},0]$ can also be generated by $S_{\vec{f}}$, according to the claim.

Note that

$$E^{\vec{f}}_{i,j}[0,0]_{n_i} E^{\vec{f}}_{s,i}[q,r] = -(-1)^{d(i)+d(s)} \sum_{v=0}^{n_i} \frac{(r+v)!}{(r+v-n_i)!} a_{i,v} E^{\vec{f}}_{s,j}[q, r+v-n_i] \quad (5.35)$$

and

$$E^{\vec{f}}_{j,i}[0,0]_{n_j-\epsilon} E^{\vec{f}}_{s,j}[q,r] = -(-1)^{d(s)+d(j)} \sum_{v=0}^{n_j} \frac{(r+v)!}{(r+v-n_j+\epsilon)!} a_{i,v} E^{\vec{f}}_{s,j}[q, r+v-n_j+\epsilon] \quad (5.36)$$

for $q, r \in \mathbb{N}$ and $\epsilon \in \{0,1\}$. Given $q \in \mathbb{N}$, we already have $E^{\vec{f}}_{s,i}[q,0]$ generated by $S_{\vec{f}}$. Hence, we can carry out induction on r to show that $E^{\vec{f}}_{s,i}[q,\mathbb{N}]$ and $E^{\vec{f}}_{s,j}[q,\mathbb{N}]$ can also be generated by $S_{\vec{f}}$. It follows that $E^{\vec{f}}_{s,i}[\mathbb{N},\mathbb{N}]$ and $E^{\vec{f}}_{s,j}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$.

Lastly, we suppose $d(s) = d(i)$. Then the fact that $E^{\vec{f}}_{s,j}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$ implies that $E^{\vec{f}}_{j,j}[\mathbb{N},\mathbb{N}]$, $E^{\vec{f}}_{j,s}[\mathbb{N},\mathbb{N}]$ and $E^{\vec{f}}_{s,s}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$, according to the claim. Symmetrically, $E^{\vec{f}}_{i,s}[\mathbb{N},\mathbb{N}]$, $E^{\vec{f}}_{i,j}[\mathbb{N},\mathbb{N}]$, $E^{\vec{f}}_{j,i}[\mathbb{N},\mathbb{N}]$ and $E^{\vec{f}}_{i,i}[\mathbb{N},\mathbb{N}]$ can also be generated by $S_{\vec{f}}$. On the other hand, if $d(s) = d(j)$, then the claim implies that $E^{\vec{f}}_{i,i}[\mathbb{N},\mathbb{N}]$, $E^{\vec{f}}_{i,s}[\mathbb{N},\mathbb{N}]$ and $E^{\vec{f}}_{s,s}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$. Since $E^{\vec{f}}_{s,i}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$. Moreover $E^{\vec{f}}_{i,s}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$ implies that $E^{\vec{f}}_{j,s}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$ (cf (5.33) and (5.34)). Using

$$E^{\vec{f}}_{i,j}[0,0]_{n_i}E^{\vec{f}}_{i,i}[q,r] = -\sum_{v=0}^{n_i} a_{i,v}\frac{(r+v)!}{(r+v-n_i)!}E^{\vec{f}}_{i,j}[q,r+v-n_i] \qquad \text{for} \quad q,r\in\mathbb{N} \tag{5.37}$$

we can show that $E^{\vec{f}}_{i,j}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$. Then the claim implies that $E^{\vec{f}}_{j,j}[\mathbb{N},\mathbb{N}]$ and $E^{\vec{f}}_{j,i}[\mathbb{N},\mathbb{N}]$ can also be generated by $S_{\vec{f}}$. Finally, if $s_1, s_2 \in \Delta_{d(j)} - \{j\}$ and $s_1 \neq s_2$, then the fact that

$$E^{\vec{f}}_{s_1,i}[0,0]_{n_i}E^{\vec{f}}_{i,s_2}[q,r] = \sum_{u=0}^{n_i}(-1)^u\frac{(u+q)!}{(u+q-n_i)!}a_{i,u}E^{\vec{f}}_{s_1,s_2}[q+u-n_1,r] \qquad \text{for} \quad q,r\in\mathbb{N} \tag{5.38}$$

show that $E^{\vec{f}}_{s_1,s_2}[\mathbb{N},\mathbb{N}]$ can be generated by $S_{\vec{f}}$.

We can use similar arguments to deal with the other case of $\vec{d} = \vec{0}$ or $k = 2$.

We take $\vec{f}$ as equation (3.23), and we define

$$\hat{R}^*(\vec{f}) = R^*(\vec{f})\bigoplus\mathbb{F}\mathbf{1} = \mathrm{Span}\{E^*_{i,j}[m,n],\mathbf{1}\,|\,i,j\in\overline{1,k}, m,n\in\mathbb{N}\} \tag{5.39}$$

where

$$E^*_{i,j}[m,n] = E^{\vec{f}}_{i,j}[m,n] - (-1)^{\iota+\epsilon(i,j)}E^{\vec{f}}_{j,i}[n,m] \qquad \text{for} \quad i,j\in\overline{1,k} \quad m,n\in\mathbb{N}. \tag{5.40}$$

We define

$$S^*_{\vec{f}} = \{E^*_{i,j}[0,0]\,|\,i\in\Delta_0,\ j\in\Delta_1\}. \tag{5.41}$$

Theorem 5.2. *When $k > 2$, the conformal superalgebra $\hat{R}^*(\vec{f})$ can be generated by $S^*_{\vec{f}}$. When $k = 2$, $\hat{R}(\vec{f})$ can be generated by $S^*_{\vec{f}}\bigcup\{E^*_{2,2}[0,1]\}$.*

Proof. This is similar to the proof of theorem 5.1. □

Let $\mathcal{C} = M_{k\times k}(\mathbb{F}) \times M_{k\times k}(\mathbb{F})$ with the grading defined in equation (4.37). Then $\mathcal{C}\bigotimes_{\mathbb{F}}\mathbb{F}[t_1,t_2]$ will be a conformal superalgebra which contains $R^{Q}(\vec{f})$ as a conformal subsuperalgebra. We define the bilinear form κ of $\mathcal{C}$ by

$$\kappa(A_{[i]},B_{[j]}) = \delta_{i+j,1}\,\mathrm{tr}(AB) \qquad \text{for} \quad A,B\in M_{k\times k}(\mathbb{F}) \quad i,j\in\mathbb{Z}_2 \tag{5.42}$$

where tr is simply the trace of the matrix. It is easy to show that κ is supersymmetric associative bilinear with grading 1.

Taking $\vec{f}$ as equation (3.8), we define

$$\hat{R}^{Q}(\vec{f}) = R^{Q}(\vec{f})\bigoplus\mathbb{F}\mathbf{1} = \mathrm{Span}\big\{\big(E^{\vec{f}}_{i,j}\big)_{[\lambda]}[m,n],\mathbf{1}\,|\,\lambda\in\mathbb{Z}_2, m,n\in\mathbb{N}\big\}. \tag{5.43}$$

We define

$$S^{Q}_{\vec{f}} = \big\{\big(E^{\vec{f}}_{i,j}\big)_{[d(i)]}[0,0],\big(E^{\vec{f}}_{j,i}\big)_{[d(j)]}[0,0]\,|\,i\in\Delta_0,\ j\in\Delta_1\big\}. \tag{5.44}$$

Theorem 5.3. *When $k > 2$ and $\vec{d} \neq \vec{0}$, the conformal superalgebra $\hat{R}^Q(\vec{f})$ is generated by $S^Q_{\vec{f}}$. When $k = 2$ or $\vec{d} = \vec{0}$, $\hat{R}^Q(\vec{f})$ can be generated by $S^Q_{\vec{f}} \bigcup \{(E^{\vec{f}}_{k,k})_{[1]}[0, 1]\}$.*

Proof. This is similar to the proof of theorem 5.1. □

Letting $c : \overline{1, 2k} \to \mathbb{Z}_2$ such that

$$c(\overline{1, k}) = 0 \qquad \text{and} \qquad c(\overline{k+1, 2k}) = 1 \tag{5.45}$$

we define $T : \overline{1, 2k} \to \overline{1, 2k}$ by

$$T(i) = \begin{cases} i + k & \text{for} \quad i \in \overline{1, k}, \\ i - k & \text{for} \quad i \in \overline{k+1, 2k}. \end{cases} \tag{5.46}$$

Let $\mathcal{C} = M_{2k\times 2k}(\mathbb{A})$ with the grading defined in equations (4.42) and (4.43). Then $\mathcal{C} \bigotimes_{\mathbb{F}} \mathbb{F}[t_1, t_2]$ is a conformal superalgebra which contains $R^\dagger(\vec{f})$ as a conformal sub-superalgebra. We define the bilinear form κ on $\mathcal{C}$ by

$$\kappa(A, B) = str(AB) \qquad \text{for} \quad A, B \in M_{2k\times 2k}(\mathbb{F}) \tag{5.47}$$

where $str : M_{2k\times 2k}(\mathbb{F}) \to \mathbb{F}$ is a linear map defined by

$$str(E_{i,j}) = (-1)^{d(i)}\delta_{i,j} \qquad \text{for} \quad i, j \in \overline{1, 2k}. \tag{5.48}$$

Then κ is supersymmetric associative bilinear with grading 0.

If we let $\mathcal{C} = M_{2k\times 2k}$ but the grading is defined as equation (4.45), then $\mathcal{C} \bigotimes_{\mathbb{F}} \mathbb{F}[t_1, t_2]$ is a conformal superalgebra which contains $R^P(\vec{f})$ as a conformal sub-superalgebra. In this case, we define the bilinear form κ on $\mathcal{C}$ by

$$\kappa(A, B) = str(AB) \qquad \text{for} \quad A, B \in M_{2k\times 2k}(\mathbb{F}) \tag{5.49}$$

where $str : M_{2k\times 2k}(\mathbb{F}) \to \mathbb{F}$ is a linear map defined by

$$str(E_{i,j}) = (-1)^{c(i)}\delta_{i,j} \qquad \text{for} \quad i, j \in \overline{1, 2k}. \tag{5.50}$$

It can be shown that κ is supersymmetric associative bilinear with grading 0.

We define

$$e_0(i) = d(i) \qquad \text{and} \qquad e_1(i) = c(i) \qquad \text{for} \quad i \in \overline{1, 2k}. \tag{5.51}$$

Taking $\vec{f}$ as equation (3.23), we define

$$\hat{R}^\dagger(\vec{f}) = R^\dagger(\vec{f}) \oplus \mathbb{F}\mathbf{1} = \mathrm{Span}\{E^\dagger_{i,j}[m, n], \mathbf{1} | i, j \in \overline{1, 2k}\, m, n \in \mathbb{N}\} \tag{5.52}$$

where for $i, j \in \overline{1, 2k}, m, n \in \mathbb{N}$,

$$E^\dagger_{i,j}[m, n] = E^{\vec{f}}_{i,j}[m, n] - (-1)^{\iota+\epsilon(i,j)+c(i)+c(j)} E^{\vec{f}}_{T(j),T(i)}[n, m]. \tag{5.53}$$

Taking $\vec{f}$ as equation (3.35), we define

$$\hat{R}^P(\vec{f}) = R^P(\vec{f}) \bigoplus \mathbb{F}\mathbf{1} = \mathrm{Span}\{E^P_{i,j}[m, n], \mathbf{1} | i, j \in \overline{1, 2k}\, m, n \in \mathbb{N}\} \tag{5.54}$$

where

$$E^P_{i,j}[m, n] = E^{\vec{f}}_{i,j}[m, n] - (-1)^{\iota+c(j)} E^{\vec{f}}_{T(j),T(i)}[n, m] \qquad \text{for} \quad i, j \in \overline{1, 2k} \quad m, n \in \mathbb{N}. \tag{5.55}$$

Letting

$$E^0_{i,j}[m, n] = E^\dagger_{i,j}[m, n] \qquad E^1_{i,j}[m, n] = E^P_{i,j}[m, n] \tag{5.56}$$

and

$$\phi_0(i, j) = \iota + \epsilon(i, j) + c(i) + c(j) \qquad \phi_1(i, j) = \iota + c(j) \tag{5.57}$$

for $i, j \in \overline{1, 2k}$ and $m, n \in \mathbb{N}$, we have

$$
\begin{aligned}
&E^l_{i,j}[p, m]_n E^l_{c,d}[q, r] \\
&\quad = \left(E^{\vec{f}}_{i,j}[p, m] - (-1)^{\phi_l(i,j)} E^{\vec{f}}_{T(j),T(i)}[m, p]\right)_n \Big(\big(E^{\vec{f}}_{c,d}[q, r] \\
&\qquad - (-1)^{\phi_l(c,d)} E^{\vec{f}}_{T(d),T(c)}[r, q]\Big) \\
&\quad = \sum_{u=0}^{n_j} (-1)^{m+u} \frac{(m+u+q)!}{(m+u+q-n)!} a_{j,u}\delta_{j,c} E^l_{i,d}[p+m+u+q-n, r] \\
&\qquad - (-1)^{G_l} \sum_{v=0}^{n_d} (-1)^p \frac{(r+v+p)!}{(r+v+p-n)!} a_{d,v}\delta_{d,i} E^l_{c,j}[q, p+m+r+v-n] \\
&\qquad - (-1)^{\phi_l(i,j)} \left(\sum_{u=0}^{n_i} (-1)^{p+u} \frac{(p+u+q)!}{(p+u+q-n)!} a_{i,u}\delta_{T(i),c} E^l_{T(j),d}\right. \\
&\qquad \times [m+p+u+q-n, r] \\
&\qquad \left. - (-1)^{G_l} \sum_{v=0}^{n_d} (-1)^m \frac{(r+v+m)!}{(r+v+m-n)!} a_{d,v}\delta_{d,T(j)} E^l_{c,T(i)}[q, m+p+r+v-n]\right) \\
&\qquad + 2(-1)^{e_l(i)} \sum_{u=0}^{n_j} \sum_{v=0}^{n_d} (q+m+u)!(r+v+p)!(-1)^{m+p+u} \\
&\qquad \times a_{j,u}a_{d,v}\delta_{j,c}\delta_{i,d}\delta_{n,p+m+u+q+r+v+1}\delta_{0,1}\mathbf{1} \\
&\qquad - 2(-1)^{\phi_l(i,j)+e_l(T(j))} \sum_{u=0}^{n_i} \sum_{v=0}^{n_d} \Big((q+p+u)!(r+v+m)!(-1)^{p+m+u} \\
&\qquad \times a_{i,u}a_{d,v}\delta_{T(i),c}\delta_{T(j),d}\delta_{n,m+p+u+q+r+v+1}\delta_{0,1}\mathbf{1}\Big)
\end{aligned} \tag{5.58}
$$

where $l \in \{0, 1\}$, $i, j, c, d \in \overline{1, k}$, $p, m, q, r \in \mathbb{N}$ and $G_l = (e_l(i) + e_l(j))(e_l(c) + e_l(d))$. We define

$$S^{\dagger}_{\vec{f}} = \left\{E^{\dagger}_{i,j}[0, 0] | i, j \in \overline{1, 2k} \text{ with } d(i) = 0, d(j) = 1\right\}. \tag{5.59}$$

Theorem 5.4. *When $k > 2$, the conformal superalgebra $\hat{R}^{\dagger}(\vec{f})$ can be generated by $S^{\dagger}_{\vec{f}}$. When $k = 2$, $\hat{R}^{\dagger}(\vec{f})$ can be generated by $S^{\dagger}_{\vec{f}} \bigcup \{E^{\dagger}_{2,2}[0, 1]\}$.*

We define

$$S^{P}_{\vec{f}} = \left\{E^{P}_{i,j}[0, 0] | i, j \in \overline{1, 2k} \text{ with } d(i) = 0, d(j) = 1\right\}. \tag{5.60}$$

In (5.58), the factor $\delta_{0,1}$ in the coefficient of $\mathbf{1}$ makes the central element of $\hat{R}^P(\vec{f})$ *de trop* and hence we develop the following theorem for $R^P(\vec{f})$ in lieu of $\hat{R}^P(\vec{f})$.

Theorem 5.5. *When $k > 2$, the conformal superalgebra $R^P(\vec{f})$ can be generated by $S^P_{\vec{f}}$. When $k = 2$, $R^P(\vec{f})$ can be generated by $S^P_{\vec{f}} \bigcup \left\{E^P_{2,2}[0, 1]\right\}$.*

Proof of theorem 6.4 and 6.5. Throughout the proof, $l \in \{0, 1\}$. Let $S^0_{\vec{f}} = S^{\dagger}_{\vec{f}}$ and $S^1_{\vec{f}} = S^P_{\vec{f}}$. Moreover, since

$$E^l_{i,j}[0, 0] = -(-1)^{\phi_l(i,j)} E^l_{T(j),T(i)}[0, 0] \qquad \text{for} \quad i, j \in \overline{1, 2k} \tag{5.61}$$

we can treat $E^l_{i,j}[0, 0]$ as an element of $S^l_{\vec{f}}$ in the rest of the calculation.

Using

$$E^l_{k,1}[0,0]_{n_1+n_k+1}E^l_{1,k}[0,0] = 2(-1)^{l+\iota+1}n_1!n_k!a_{1,n_1}a_{k,n_k}\mathbf{1} \tag{5.62}$$

we know $\mathbf{1}$ can be generated by $S^l_{\vec{f}}$ and we can ignore the extension part in equation (5.58) for the rest of our proof.

□

Claim: If $E^l_{a,b}[\mathbb{N},\overline{0,r}]$ can be generated by $S^l_{\vec{f}}$ or $S^l_{\vec{f}}\bigcup\left\{E^l_{k,k}[0,1]\right\}$ for $r\in\mathbb{N}, a,b\in\overline{1,2k}$ with $d(a)\neq d(b)$, then $E^l_{b,b}[\mathbb{N},\overline{0,r}]$, $E^l_{T(b),b}[\mathbb{N},\overline{0,r}]$, $E^l_{b,a}[\mathbb{N},\overline{0,r}]E^l_{a,a}[\mathbb{N},\overline{0,r}]$ and $E^l_{T(a),a}[\mathbb{N},\overline{0,r}]$ can also be generated by $S^l_{\vec{f}}$ or $S^l_{\vec{f}}\bigcup\left\{E^l_{k,k}[0,1]\right\}$, respectively.

To prove the claim, we employ its hypothesis. Note that

$$\begin{aligned}
&E^l_{c,a}[0,0]_{(n_a+n_b+r'+1)}E^l_{a,b}[q+n_b+r'+1,r'] \\
&\quad = \sum_{u=0}^{n_a}(-1)^u\frac{(u+q+n_b+r'+1)!}{(u+q-n_a)!}a_{a,u}E^l_{c,b}[u+q-n_a,r']
\end{aligned} \tag{5.63}$$

where c can be either b or $T(b)$ for $q,r'\in\mathbb{N}$.

Hence, we can carry out induction on q for different $r'\leqslant r$ to show that both $E^l_{b,b}[\mathbb{N},\overline{0,r}]$ and $E^l_{T(b),b}[\mathbb{N},\overline{0,r}]$ can be generated by $S^l_{\vec{f}}$ or $S^l_{\vec{f}}\bigcup\left\{E^l_{k,k}[0,1]\right\}$, respectively. On the other hand,

$$E^l_{b,a}[0,0]_{n_b}E^l_{b,b}[q,r'] = -\sum_{v=0}^{n_b}a_{b,v}\frac{(r+v)!}{(r+v-n_b)!}E^*_{b,a}[q,r'+v-n_b] \qquad \text{for} \quad q,r'\in\mathbb{N}. \tag{5.64}$$

Hence, we can perform an induction on $r'\leqslant r$ for different $q\in\mathbb{N}$ to show that $E^l_{b,a}[\mathbb{N},\overline{0,r}]$ can be generated by $S^l_{\vec{f}}$ or $S^l_{\vec{f}}\bigcup\left\{E^l_{k,k}[0,1]\right\}$, respectively. This implies that $E^l_{a,a}[\mathbb{N},\overline{0,r}]$ and $E^l_{T(a),a}[\mathbb{N},\overline{0,r}]$ can also be generated by $S^l_{\vec{f}}$ or $S^l_{\vec{f}}\bigcup\left\{E^l_{k,k}[0,1]\right\}$, respectively.

Now we can divide the proof into two cases:

Case: $\vec{d}\neq\vec{0}$ and $k>2$

There exist $j\in\overline{1,2k}$ such that $n_j\neq 0$. We pick i such that $d(i)=d(j)+1$. Then for $s\in\overline{1,2k}-\{i,j,T(i),T(j)\}\neq\emptyset$,

$$E^l_{i,j}[0,0]_{n_j-\epsilon}E^l_{j,s}[q,r] = \sum_{u=0}^{n_j}(-1)^u\frac{(u+q)!}{(u+q-n_j+\epsilon)!}a_{j,u}E^l_{i,s}[u+q-n_j+\epsilon,r] \tag{5.65}$$

$$E^l_{j,i}[0,0]_{n_i}E^l_{i,s}[q,r] = \sum_{u=0}^{n_i}(-1)^u\frac{(u+q)!}{(u+q-n_i)!}a_{i,u}E^l_{j,s}[u+q-n_j,r] \tag{5.66}$$

where $r,q\in\mathbb{N}$ and $\epsilon\in\{0,1\}$. Given $r\in\mathbb{N}$ (5.65) with $\epsilon=0$ tells us that if $E^l_{j,s}[0,r]$ can be generated by $S^l_{\vec{f}}$, then so does $E^l_{i,s}[0,r]$, while equation (5.66) implies the opposite. Since either $E^l_{j,s}[0,0]$ or $E^l_{i,s}[0,0]$ belongs to $S^l_{\vec{f}}$, both can be generated by $S^l_{\vec{f}}$. We suppose $E^l_{j,s}[q',r]$ and $E^l_{i,s}[q',r]$ can be generated by $S^l_{\vec{f}}$ for all $q'\leqslant q$ for $q\in\mathbb{N}$. Equation (5.65) with $\epsilon=1$ implies that $E^l_{i,s}[q+1,r]$ can be generated by $S^l_{\vec{f}}$ and then equation (5.66) implies $E^l_{j,s}[q+1,r]$ does also. As a result, $E^l_{j,s}[\mathbb{N},0]$ and $E^l_{i,s}[\mathbb{N},0]$ can be generated by $S^l_{\vec{f}}$.

Next we would like to show that $E^l_{s,i}[l,0]$ can be generated by $S^l_{\vec{f}}$ for all $l \in \mathbb{N}$. If $d(s)=d(i)$, then $E^l_{s,i}[\mathbb{N},0]$ can be generated by $S^l_{\vec{f}}$ because of the symmetry between i and s. On the other hand, if $d(s)=d(j)$, then the fact that $E^l_{i,s}[\mathbb{N},0]$ can be generated by $S^l_{\vec{f}}$ implies that $E^l_{s,i}[\mathbb{N},0]$ can also be generated by $S^l_{\vec{f}}$, according to the claim.

Note that

$$E^l_{i,j}[0,0]_{n_i}E^l_{s,i}[q,r]=-(-1)^{(e_l(i)+e_l(j))(e_l(s)+e_l(i))}\sum_{v=0}^{n_i}\frac{(r+v)!}{(r+v-n_i)!}a_{i,v}E^l_{s,j}[q,r+v-n_i] \tag{5.67}$$

$$\begin{aligned}E^l_{j,i}[0,0]_{n_j-\epsilon}&E^l_{s,j}[q,r]\\ &=-(-1)^{(e_l(j)+e_l(i))(e_l(s)+e_l(j))}\sum_{v=0}^{n_j}\frac{(r+v)!}{(r+v-n_j+\epsilon)!}a_{i,v}E^l_{s,i}[q,r+v-n_j+\epsilon]\end{aligned} \tag{5.68}$$

for $q,r\in\mathbb{N}$ and $\epsilon\in\{0,1\}$. Given $q\in\mathbb{N}$, we already have $E^l_{s,i}[q,0]$ or $E^l_{s,j}[q,0]$ generated by $S^l_{\vec{f}}$. Hence, we can carry out an induction on r to show that $E^l_{s,i}[q,\mathbb{N}]$ and $E^l_{s,j}[q,\mathbb{N}]$ can also be generated by $S^l_{\vec{f}}$. It follows that $E^l_{s,i}[\mathbb{N},\mathbb{N}]$ and $E^l_{s,j}[\mathbb{N},\mathbb{N}]$ can be generated by $S^l_{\vec{f}}$.

Finally, suppose $d(s)=d(i)$. Then the fact that $E^l_{s,j}[\mathbb{N},\mathbb{N}]$ can be generated by $S^l_{\vec{f}}$ implies that $E^l_{j,j}[\mathbb{N},\mathbb{N}]$, $E^l_{T(j),j}[\mathbb{N},\overline{0,r}]$, $E^l_{s,s}[\mathbb{N},\mathbb{N}]$ and $E^l_{T(s),s}[\mathbb{N},\mathbb{N}]$ can be generated by $S^l_{\vec{f}}$, according to the claim. Symmetrically, $E^l_{i,j}[\mathbb{N},\mathbb{N}]$, $E^l_{i,i}[\mathbb{N},\mathbb{N}]$ and $E^l_{T(i),i}[\mathbb{N},\mathbb{N}]$ can also be generated by $S^l_{\vec{f}}$. On the other hand, if $d(s)=d(j)$, then the fact that $E^l_{s,i}[\mathbb{N},\mathbb{N}]$ can be generated by $S^l_{\vec{f}}$ implies that $E^l_{i,i}[\mathbb{N},\mathbb{N}]$, $E^l_{T(i),i}[\mathbb{N},\mathbb{N}]$, $E^l_{i,s}[\mathbb{N},\mathbb{N}]$, $E^l_{s,s}[\mathbb{N},\mathbb{N}]$ and $E^l_{T(s),s}[\mathbb{N},\mathbb{N}]$ can also be generated by $S^l_{\vec{f}}$, according to the claim. Moreover, the fact that $E^l_{i,s}[\mathbb{N},\mathbb{N}]$ can be generated by $S^l_{\vec{f}}$ implies that $E^l_{j,s}[\mathbb{N},\mathbb{N}]$ can also be generated by $S^l_{\vec{f}}$ (cf equations (5.65) and (5.66)). Using

$$E^l_{i,j}[0,0]_{n_i}E^l_{i,i}[q,r]=-\sum_{v=0}^{n_i}a_{i,v}\frac{(r+v)!}{(r+v-n_i)!}E^l_{i,j}[q,r+v-n_i]\qquad\text{for}\quad q,r\in\mathbb{N} \tag{5.69}$$

we can show that $E^l_{i,j}[\mathbb{N},\mathbb{N}]$ can be generated by $S^l_{\vec{f}}$. Then the claim will imply that $E^l_{j,j}[\mathbb{N},\mathbb{N}]$, $E^l_{T(j),j}[\mathbb{N},\mathbb{N}]$ and $E^l_{j,i}[\mathbb{N},\mathbb{N}]$ can also be generated by $S^l_{\vec{f}}$. Finally, if $d(s_1)=d(s_2)=j$ with s_1 not equal to either s_2 or $T(s_2)$, then

$$E^{\vec{f}}_{s_1,i}[0,0]_{n_i}E^{\vec{f}}_{i,s_2}[q,r]=\sum_{u=0}^{n_i}(-1)^u\frac{(u+q)!}{(u+q-n_i)!}a_{i,u}E^{\vec{f}}_{s_1,s_2}[q+u-n_1,r]\qquad\text{for}\quad q,r\in\mathbb{N} \tag{5.70}$$

show that $E^{\vec{f}}_{s_1,s_2}[\mathbb{N},\mathbb{N}]$ can be generated by $S^l_{\vec{f}}$.

Again, we can use similar arguments to deal with the case of $\vec{d}=\vec{0}$ or $k=2$.

Taking $\vec{f}$ as equation (3.35), we let $m\in\overline{1,k}$. We define $e_2:\overline{1,m+2k}\to\mathbb{Z}_2$ by

$$e_2(i)=\begin{cases}0 & \text{if}\quad i\in\overline{1,m}\\ 1 & \text{if}\quad i\in\overline{m+1,m+2k}.\end{cases} \tag{5.71}$$

Letting $\mathcal{C} = M_{m+2k,m+2k}(\mathbb{F})$ with the grading defined as equation (4.47), then $\mathcal{C}\bigotimes_{\mathbb{F}}[t_1, t_2]$ is a conformal superalgebra which contains $R^{osp}(\vec{f})$ as a conformal sub-superalgebra. We define the bilinear form κ on $\mathcal{C}$ by

$$\kappa(A, B) = str(AB) \qquad \text{for} \quad A, B \in M_{m+2k,m+2k} \tag{5.72}$$

where $str : M_{m+2k,m+2k}(\mathbb{F}) \to \mathbb{F}$ is a linear map defined by

$$str(E_{i,j}) = (-1)^{e_2(i)}\delta_{i,j} \qquad \text{for} \quad i, j \in \overline{1, m+2k}. \tag{5.73}$$

Then, κ is a supersymmetric associative bilinear form with grading 0.

We define

$$\begin{aligned}\hat{R}^{osp}(\vec{f}) = R^{osp}(\vec{f}) \bigoplus \mathbb{F}\mathbf{1} = \operatorname{Span}\{&A_{\alpha,\beta}[q, r]\\ &F_{\alpha,i+m}[q, r], H_{i+m,j+m}[q, r], \mathbf{1} | i, j \in \overline{1, 2k}, \alpha, \beta \in \overline{1, m}, q, r \in \mathbb{N}\}\end{aligned} \tag{5.74}$$

where

$$A_{\alpha,\beta}[q, r] = E^{\vec{f}}_{\alpha,\beta}[q, r] - (-1)^{\iota} E^{\vec{f}}_{\beta,\alpha}[r, q] \tag{5.75}$$

$$F_{\alpha,i+m}[q, r] = E^{\vec{f}}_{\alpha,i+m}[q, r] + (-1)^{\iota+c(i)} E^{\vec{f}}_{T(i)+m,c}[r, q] \tag{5.76}$$

$$H_{i+m,j+m}[q, r] = E^{\vec{f}}_{i+m,j+m}[q, r] - (-1)^{\iota+c(i)+c(j)} E^{\vec{f}}_{T(j)+m,T(i)+m}[r, q] \tag{5.77}$$

with $\alpha, \beta \in \overline{1, m}$, $i, j \in \overline{1, 2k}$ and $q, r \in \mathbb{N}$. Then we have

$$A_{\alpha,\beta}[p, w]_n H_{i+m,j+m}[q, r] = H_{i+m,j+m}[q, r]_n A_{\alpha,\beta}[p, w] = 0 \tag{5.78}$$

$$\begin{aligned}A_{\alpha,\beta}[p, w]_n A_{\gamma,\theta}[q, r] = &\sum_{u=0}^{n_\beta}(-1)^{w+u}\frac{(w+u+q)!}{(w+u+q-n)!}a_{\beta,u}\delta_{\beta,\gamma}A_{\alpha,\theta}[p+w+u+q-n, r]\\ &- \sum_{v=0}^{n_\theta}(-1)^{p}\frac{(r+v+p)!}{(r+v+p-n)!}a_{\theta,v}\delta_{\theta,\alpha}A_{\gamma,\beta}[q, p+w+r+v-n]\\ &- (-1)^{\iota}\left(\sum_{u=0}^{n_\alpha}(-1)^{p+u}\frac{(p+u+q)!}{(p+u+q-n)!}a_{\alpha,u}\delta_{\alpha,\gamma}A_{\beta,\theta}[w+p+u+q-n, r]\right.\\ &\left.- \sum_{v=0}^{n_\theta}(-1)^{w}\frac{(r+v+w)!}{(r+v+w-n)!}a_{\theta,v}\delta_{\theta,\beta}A_{\gamma,\alpha}[q, w+p+r+v-n]\right)\\ &+ 2\sum_{u=0}^{n_\beta}\sum_{v=0}^{n_\theta}(q+w+u)!(r+v+p)!(-1)^{w+p+\iota}a_{\beta,u}a_{\theta,v}\delta_{\beta,\gamma}\delta_{\alpha,\theta}\delta_{n,p+w+u+q+r+v+1}\mathbf{1}\\ &- 2\sum_{u=0}^{n_\alpha}\sum_{v=0}^{n_\theta}(q+p+u)!(r+v+w)!(-1)^{p+w}a_{\alpha,u}a_{\theta,v}\delta_{\alpha,\gamma}\delta_{\beta,\theta}\delta_{n,w+p+u+q+r+v+1}\mathbf{1}\end{aligned} \tag{5.79}$$

$$\begin{aligned}&F_{\alpha,j+m}[p, w]_n F_{\beta,d+m}[q, r]\\ &\quad = (-1)^{c(d)+w}\sum_{u=0}^{n_j}\frac{(w+u+r)!}{(w+u+r-n)!}a_{j,u}\delta_{j,T(d)}A_{\alpha,\beta}[p+w+u+r-n, q]\\ &\quad + (-1)^{\iota+c(d)+p}\sum_{v=0}^{n_\beta}\frac{(q+v+p)!}{(q+v+p-n)!}a_{\beta,v}\delta_{\beta,\alpha}H_{T(d)+m,j+m}[r, p+w+q+v-n]\end{aligned}$$

$$+(-1)^{c(d)+w+p}2\sum_{u=0}^{n_j}\sum_{v=0}^{n_\beta}(r+w+u)!(q+v+p)!a_{j,u}a_{\beta,v}\delta_{j,T(d)} \times \delta_{\alpha,\beta}\delta_{n,p+w+q+r+v+1}\mathbf{1} \tag{5.80}$$

$$\begin{aligned}
&H_{i+m,j+m}[p,w]_n H_{c+m,d+m}[q,r]\\
&\quad=\sum_{u=0}^{n_j}(-1)^{w+u}\frac{(w+u+q)!}{(w+u+q-n)!}a_{j,u}\delta_{j,c}H_{i+m,d+m}[p+w+u+q-n,r]\\
&\quad-\sum_{v=0}^{n_d}(-1)^{p}\frac{(r+v+p)!}{(r+v+p-n)!}a_{d,v}\delta_{d,i}H_{c+m,j+m}[q,p+w+r+v-n]\\
&\quad-(-1)^{\iota+c(i)+c(j)}\left(\sum_{u=0}^{n_i}(-1)^{p+u}\frac{(p+u+q)!}{(p+u+q-n)!}a_{i,u}\delta_{T(i),c}H_{T(j)+m,d+m}\right.\\
&\quad\times[w+p+u+q-n,r]\\
&\quad\left.-\sum_{v=0}^{n_d}(-1)^{w}\frac{(r+v+w)!}{(r+v+w-n)!}a_{d,v}\delta_{d,T(j)}H_{c+m,T(i)+m}[q,w+p+r+v-n]\right)\\
&\quad-2\sum_{u=0}^{n_j}\sum_{v=0}^{n_d}(q+w+u)!(r+v+p)!(-1)^{w+p+u}a_{j,u}a_{d,v}\delta_{j,c}\delta_{i,d}\delta_{n,p+w+u+q+r+v+1}\mathbf{1}\\
&\quad+2(-1)^{\iota+c(i)+c(j)}\sum_{u=0}^{n_i}\sum_{v=0}^{n_d}((q+p+u)!(r+v+w)!(-1)^{p+w+u}\\
&\quad\times a_{i,u}a_{d,v}\delta_{T(i),c}\delta_{T(j),d}\delta_{n,w+p+u+q+r+v+1}\mathbf{1})
\end{aligned}\tag{5.81}$$

$$\begin{aligned}
&A_{\alpha,\beta}[q,r]_n F_{\gamma,j+m}[p,w]\\
&\quad=(-1)^{r+\iota}\sum_{u=0}^{n_\beta}\frac{(r+u+p)!}{(r+u+p-n)!}a_{\beta,u}\delta_{\beta,\gamma}F_{\alpha,j+m}[q+r+u+p-n,w]\\
&\quad+(-1)^{q+1}\sum_{u=0}^{n_\alpha}\frac{(q+u+p)!}{(q+u+p-n)!}a_{\alpha,u}\delta_{\alpha,\gamma}F_{\beta,j+m}[r+q+u+p-n,w]
\end{aligned}\tag{5.82}$$

$$\begin{aligned}
&H_{i+m,j+m}[p,w]_n F_{\alpha,d+m}[q,r]\\
&\quad=(-1)^{p+1}\sum_{v=0}^{n_d}\frac{(r+v+p)!}{(r+v+p-n)!}a_{d,v}\delta_{d,i}F_{\alpha,j+m}[q,p+w+r+v-n]\\
&\quad+(-1)^{\iota+c(i)+c(j)+w}\sum_{v=0}^{n_d}\frac{(r+v+w)!}{(r+v+w-n)!}a_{d,v}\delta_{d,T(j)}F_{\alpha,T(i)+m}\\
&\quad\times[q,w+p+r+v-n]
\end{aligned}\tag{5.83}$$

$$\begin{aligned}
&F_{\alpha,j+m}[p,w]_n A_{\beta,\gamma}[q,r]\\
&\quad=(-1)^{p}\sum_{v=0}^{n_\gamma}\frac{(r+v+p)!}{(r+v+p-n)!}a_{\gamma,v}\delta_{\gamma,\alpha}F_{\beta,j+m}[q,p+w+r+v-n]\\
&\quad+(-1)^{\iota+p}\sum_{v=0}^{n_\beta}\frac{(q+v+p)!}{(q+v+p-n)!}a_{\beta,v}\delta_{\beta,\alpha}F_{\gamma,j+m}[r,p+w+q+v-n]
\end{aligned}\tag{5.84}$$

and

$$F_{\alpha,j+m}[p,w]_n H_{c+m,d+m}[q,r]$$
$$= (-1)^{w+\iota}\sum_{u=0}^{n_j}\frac{(w+u+q)!}{(w+u+q-n)!}a_{j,u}\delta_{j,c}F_{\alpha,d+m}[p+w+u+q-n,r]$$
$$-(-1)^{c(c)+c(d)+w}\sum_{u=0}^{n_j}\frac{(w+u+r)!}{(w+u+r-n)!}a_{j,u}\delta_{j,T(d)}F_{\alpha,T(c)+m}$$
$$\times[p+w+u+r-n,q] \tag{5.85}$$

for $\alpha,\beta,\gamma,\theta\in\overline{1,m}$, $i,j,c,d\in\overline{1,2k}$ and $p,w,q,r\in\mathbb{N}$. Define

$$S_{\vec{f}}^{osp}=\{F_{\alpha,i+m}[0,0]|\alpha\in\overline{1,m},\ i\in\overline{1,2k}\}. \tag{5.86}$$

Then we can use equations (5.78)–(5.85) to prove the following theorem:

Theorem 5.6. *When $\vec{d}\neq\vec{0}$, then $\hat{R}^{osp}(\vec{f})$ can be generated by $S_{\vec{f}}^{osp}$. Otherwise, it can be generated by $S_{\vec{f}}^{osp}\bigcup\{H_{k+m,k+m}[0,1]\}$.*

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